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Nonsymmetric Kaluza–Klein and Jordan–Thiry Theory in a General Non-Abelian Case

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This paper is devoted to an $(n+4)$-dimensional unification of NGT (nonsymmetric gravitation theory) and Yang–Mills theory in a Jordan–Thiry manner. We find "interference effects" between gravitational and Yang–Mills fields which appear to be due to the skew-symmetric part of the metric on the $(n+4)$-dimensional manifold (nonsymmetrically metrized principal fiber bundle). Our unification, called the nonsymmetric-non-Abelian Jordan–Thiry theory, becomes classical if the skew-symmetric part of the metric is zero. We find the Yang–Mills field Lagrangian up to the second order of approximation in $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. We also deal with the Lagrangian for the scalar field (connected to the "gravitational constant"). We consider the spin content of the theory and a relationship between the cosmological constant and the coupling constant between the skewon field and the gauge field in the first order of approximation. We show how to derive a dielectric model of a confinement from "interference effects" in these theories. We underline some similarities between the nonsymmetric Jordan–Thiry Lagrangian in the flat space limit and the soliton bag model Lagrangian.

INTRODUCTION

The aim of this paper is to construct the Kaluza–Klein (Jordan–Thiry) analogue with Einstein's geometry on a principal fiber bundle in the general non-Abelian case (for classical results see refs. 1–91). In other words, it will be an $(n+4)$-dimensional unification of NGT (nonsymmetric gravitation theory), gauge (Yang–Mills) fields, and scalar forces connected to the gravitational constant (as in the scalar-tensor theories of gravitation; see ref. 50). Our unification uses a nonsymmetric metrization of fiber bundles. We introduce a scalar field ρ in a Jordan–Thiry manner (see ref. 21). We get the following "interference effects" between Yang–Mills and gravitational fields:

1. A new term in the Yang–Mills Lagrangian,

$$\frac{-1}{4\pi} h_{ab}[g^{[\mu\nu]}H^{a}{}_{\mu\nu}][g^{[\alpha\beta]}H^{b}{}_{\alpha\beta}]$$

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2. A change in the classical part of the Yang-Mills Lagrangian, replacing h_{ab} by

$$l_{ab} = h_{ab} + \mu k_{ab}$$

3. The existence of a Yang-Mills field polarization of the vacuum $M^{a}{}_{\alpha\beta}$ with an interpretation as a torsion in higher dimensions.

4. An additional term in the Kerner-Wong equation (equation of motion for a test particle in the gravitational and Yang-Mills fields)

$$\frac{1}{2}\left(\frac{q^b}{m_0}\right)(l_{bd}g^{\alpha\beta} - l_{db}g^{\alpha\beta})L^{d}{}_{\beta\gamma}u^{\gamma}$$

where m_0 is the rest mass of a test particle and q^b is its color (isotopic) charge.

5. A new energy-momentum tensor $T^{\text{gauge}}_{\alpha\beta}$ with zero trace.

6. Sources for Yang-Mills fields, the current $\mathbf{j}^{\alpha a}$.

All of these effects vanish if the metric on $\underline{P}$ (fiber bundle becomes symmetric. In this case we get the classical results.

We get in the Moffat-Ricci curvature scalar, on an $(n+4)$-dimensional manifold $\underline{P}$, a Lagrangian of the scalar field Ψ,

$$\mathscr{L}_{\text{scal}}(\Psi) = (m\tilde{g}^{(\gamma\nu)} + n^2 g^{[\mu\nu]}g_{\delta\mu}\tilde{g}^{(\delta\gamma)})\Psi_{,\nu}\Psi_{,\gamma}$$

where

$$m = (l^{[dc]}l_{[dc]} - 3n(n-1)), \qquad n = \dim G$$

This field is connected to the gravitational constant by $K = e^{-(n+2)\Psi}$, where K is the gravitational constant. The trace of the energy-momentum tensor for this field is not zero. This suggests that Ψ is massive and has Yukawa-type behavior. This indicates that Ψ has a short range and the theory does not violate the weak equivalence principle. Furthermore, the gravitational "constant" K does not change at long distances. This statement also supports the masslike term in the equation for Ψ,

$$-8(n+2)\pi\, e^{-(n+2)\Psi}(\mathscr{L}_{YM} - 2\Phi) = -8(n+2)\pi\, e^{-(n+2)\Psi}\mathscr{L}_{YM} - e^{(n+2)\Psi}A(\mu)$$

where

$$\mathscr{L}_{YM} = \frac{-l_{ab}}{8\pi}[2(g^{[\mu\nu]}H^{a}{}_{\mu\nu})(g^{[\alpha\beta]}H^{b}{}_{\alpha\beta}) - L^{a\alpha\beta}H^{b}{}_{\alpha\beta}]$$

is the Lagrangian for the Yang-Mills field; Φ has an interpretation as a cosmological term in our theory

$$\Phi(\mu) = \frac{e^{2(n+2)\Psi}}{16\pi}A(\mu) \sim (e^{2(n+2)\Psi})\frac{\text{const}}{\mu}$$

$$\text{or} \quad \sim(e^{2(n+2)\Psi}) \qquad (\text{for large } \mu)$$

which now depends on the scalar field Ψ. We also get a scalar-force term in the equation of motion for a charged test particle moving in the gravitational and Yang–Mills fields:

$$-\frac{\|q\|^2}{4m_0^2}\tilde{g}^{(\alpha\beta)}\, e^{2\Psi}\Psi_{,\beta}$$

where $\|q\|^2 = -h_{ab}q^a q^b$ is the length squared of the color (isotopic) charge of a test particle. This force is of short range. If the skew-symmetric part of the metric γ_{AB} becomes zero, most of these effects vanish. However, the propagation of the scalar field is possible only if $n \geq 2$.

Let us make some remarks on differences between the nonsymmetric non-Abelian Kaluza–Klein and Jordan–Thiry theories. In the nonsymmetric non-Abelian Kaluza–Klein theory there is an Ansatz $\rho = 1$ ($\gamma_{ab} = l_{ab} = h_{ab} + \mu k_{ab}$). This condition seems to be quite arbitrary and because of this we consider a more general case called Jordan–Thiry theory where $\gamma_{ab} = \rho^2(h_{ab} + \mu k_{ab})$ and $\rho = \rho(x)$ is a dynamical field.

Moreover, the detailed examination of geodetic equations in both cases reveals the following. If $\rho = \text{const}$, the geodetic equation possesses an integral of motion

$$\gamma(\text{hor}(u(\tau)), \text{hor}(u(\tau))) = g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \text{const} \qquad (**)$$

which allows us to maintain an initial normalization of the four-velocity of a test particle. In the case with $\gamma_{ab}(x) = \rho^2 l_{ab}$ (i.e., $\rho \neq \text{const}$) this is not possible in general (see Section 4.12). For this the condition $\gamma_{ab} = l_{ab}$ does not seem to be an Ansatz in the theory, but rather a conclusion from (**).

This paper is organized as follows. In Section 1 we give some elements of geometry used in the paper. Section 2 is devoted to a nonsymmetric tensor on a Lie group. In Section 3 we present a nonsymmetric metrization of the fiber bundle. In Section 4 we formulate the nonsymmetric Jordan–Thiry theory in a general non-Abelian case. We calculate connections $\omega^A{}_B$ and $W^A{}_B$ on the $(n+4)$-dimensional manifold which are analogous to the connections $\bar{\omega}^\alpha{}_\beta$ and $\bar{W}^\alpha{}_\beta$ from NGT. In Section 5 we write the geodetic equation on $\underline{P}$ (nonsymmetrically metrized fiber bundle with scalar field ρ) and we find new corrections to the equation of motion for a test particle. In Section 6 we calculate the 2-form of torsion for the connection $\omega^A{}_B$ and the 2-form of curvature for $\omega^A{}_B$. We calculate also the curvature tensor for $\omega^A{}_B$ and $W^A{}_B$. After this we find the Moffat–Ricci curvature scalar for $W^A{}_B$ which plays the role of the Lagrangian in our theory. In Section 7 we deal with a connection $\tilde{\omega}^a{}_b$ on a typical fiber and with the cosmological constant in our theory. In Section 8 we perform a conformal transformation for the $g_{\mu\nu}$ tensor and we transform the scalar field ρ to Ψ. Section 9 is devoted to the gauge invariance of the Lagrangian. In Section 10 we define the Palatini variational principle for $R(W)$ and we get equations for the gravitational, Yang–Mills, and scalar fields. We interpret our results.

Section 11 is devoted to some special cases in our theory. In Section 12 we deal with the linearization procedure in our theory. Section 13 deals with geodetic equations in a linear approximation. In Section 14 we examine some general properties of geodesics in our theory and the geodetic deviation equation. Section 15 gives some conclusions and prospects.

In Appendix A we consider a more general case for

$$\gamma_{ab} = P_{ab}(p) = P_{ab}(x, g)$$

$$p \in \underline{P}, \qquad x \in U \subset E, \qquad g \in G$$

where P_{ab} is a dynamical field depending on a space-time point $(x \in E)$ and right-invariant with respect to the (right) action of the group G. The detailed examination of geodetic equations on $\underline{P}$ leads to the conclusion that the tensor $\hat{P}$ has the shape

$$P_{ab}(x, g) = \rho^2(x) l_{ab}(g) \qquad \text{or} \qquad \hat{P} = \rho^2 l$$

where l_{ab} is right-invariant with respect to the (right) action of the group G (i.e., it has a factorization property), and $\rho = \rho(x)$ is a scalar field on E (i.e., $\hat{P} = P_{ab}\theta^a \otimes \theta^b$, $l = l_{ab}\theta^a \otimes \theta^b$). Moreover, in order to get a proper limit for the Yang–Mills Lagrangian, i.e., for $\mu = 0$, we suppose that $l_{(ab)} = h_{(ab)}$ (the bi-invariant tensor on G).

Moreover, we can get right-invariance of l_{ab}, demanding the gauge invariance of the curvature scalar built from a connection $\omega^A{}_B$ (i.e., we come to the nonsymmetric metrization of the fiber bundle P considered in Section 3.

We can summarize our conclusions in two theorems.

Theorem I:

1. Let $\underline{P}$ be a principal fiber bundle over a space-time E with a structural group G (semisimple and compact), a projection π, and let us define on $\underline{P}$ a connection ω.

2. Let $\bar{\omega}^\alpha{}_\beta$ be a linear connection on a fiber bundle of frames over E compatible in the Einstein–Kaufman sense with the nonsymmetric (real) tensor $g_{\alpha\beta}$ defined on E.

3. Let P_{ab} be a family of nonsymmetric right-invariant tensor fields defined on G and parametrized by a point on E, i.e., $P_{ab} = P_{ab}(x, g)$, $x \in U \subset E$, $g \in G$ $(\hat{P} = P_{ab}\theta^a \otimes \theta^b)$.

4. Let $\gamma = \gamma_{AB}\theta^A \otimes \theta^B$ be a tensor field on $\underline{P}$ (a nonsymmetric metric) and let $\omega^A{}_B$ be a linear connection on a fiber bundle of frames over $\underline{P}$ compatible in the Einstein–Kaufman sense with this tensor (i.e., we have

to do with $\tilde{\omega} = \omega^A{}_B x^B{}_A$). The tensor γ in a lift horizontal basis has the form

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & P_{ab} \end{array}\right)$$

i.e.,

$$\underline{\gamma} = \gamma_{(AB)}\theta^A \otimes \theta^B = \pi^* \underline{g} \oplus P_{(ab)}\theta^a \otimes \theta^b = \pi^* \underline{g} \oplus \underline{\hat{P}}$$
$$\bar{\gamma} = \gamma_{[AB]}\theta^A \wedge \theta^B = \pi^* \bar{g} \oplus P_{[ab]}\theta^a \wedge \theta^b = \pi^* \bar{g} \oplus \bar{\hat{P}}$$

(see Section 3 for some details concerning our notations).

5. Let geodetic equations (for a geodetic Γ) with respect to the connection $\omega^A{}_B$ possess n first integrals of motion $v^\alpha = \text{const}$ being Ad-type quantities and a linear function of

$$u^a = \frac{dx^a}{dt}, \qquad u^a = (\text{ver}(u))^a$$

(where u is a tangent vector field to geodesic Γ), i.e., there is a bi-invariant invertible matrix $\varkappa^a{}_b$ such that

$$v^a = \varkappa^a{}_b u^b$$

or there is a bi-invariant invertible linear transformation field on $\underline{P}$ such that

$$\hat{\varkappa} : \text{Ver}(\text{Tan}_p(P)) \to \text{Ver}(\text{Tan}_p(P))$$

$\hat{\varkappa}_p \circ \omega_p = \omega_p \circ \hat{\varkappa}_p$, and $v_p = \hat{\varkappa}_p(\text{ver}(u_p))$, $p \in U \times G$, $U \subset E$ in a local trivialization, and $v_p = \text{const}$ for $\Gamma \subset P$. Then, there exists a scalar field $\rho = \rho(x)$ and a nonsymmetric tensor l_{ab} on G such that:

1. $P_{ab} = \rho^2 l_{ab}$ (a factorization property), ($\hat{P} = \rho^2 l$).
2. l_{ab} is right-invariant with respect to the right action of the group G on G (see Section 2 for details and definitions).
3. γ and $\tilde{\omega}$ are right-invariant with respect to the right action of the group G on P, i.e., $\hat{\phi}'(g)\gamma = \gamma$ and $\hat{\phi}^*(g)\tilde{\omega} = \tilde{\omega}$, where $\hat{\phi}$ is an action of the group G on the bundle of frames over $(\underline{P}, \gamma)$ lifted from the gauge bundle.

Theorem II. Let conditions 1–5 be satisfied and in addition:

6. Let geodetic equations for a curve Γ with respect to $\omega^A{}_B$ possess a first integral of motion

$$g_{(\alpha\beta)} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = \text{const}$$

on Γ, i.e., $\gamma(\text{hor}(u), \text{hor}(u)) = \text{const}$, where u is tangent to $\Gamma \subset \underline{P}$. Then the scalar field $\rho(x) = \text{const}$, i.e., $P_{ab} = l_{ab}$ ($\hat{P} = l$) (up to a constant factor which can be absorbed in the definition of l_{ab}) [hor is understood in the sense of ω on P (a gauge bundle)]. For l_{ab} right-invariant and demanding a proper limit for the Yang–Mills Lagrangian in the case $\mu = 0$, we easily get that

$$l_{ab} = h_{ab} + \mu k_{ab} \qquad (l = h + \mu k)$$

where h_{ab} is a bi-invariant tensor on G and k_{ab} is a skew-symmetric right-invariant tensor on the group G defined in Section 2.

Briefly, if conditions 1–6 are satisfied, we get the nonsymmetric non-Abelian Kaluza–Klein theory (N^2AK^2T). If conditions 1–5 are satisfied, we get the nonsymmetric non-Abelain Jordan–Thiry theory (N^2AJT^2).

In this way the assumption of the factorization property of P_{ab} and the constancy of a field ρ do not seem to be arbitrary conditions, but rather the conclusions of Theorems I and II. The proofs of Theorems I and II can be found in Appendix A. Both conclusions justify our interest in the theory presented in this paper from the physical and mathematical points of view.

Let us note that our construction with a right-invariant l_{ab}, (l) tensor leads to a notion which can be called the Einstein–Kaufman G-structure (right G-structure).

In Appendix B we consider some problems connected to test particle motion on a nonsymmetrically metrized bundle $\underline{P}$.

1. ELEMENTS OF GEOMETRY

In this section we introduce the notations and define the geometric quantities used in this paper. We use a smooth principal fiber bundle $\underline{P}$, which includes in its definition the following list of differentiable manifolds and smooth maps:

A total (bundle) space $\underline{P}$.

A base space E; in our case it is a space-time.

A projection $\pi : P \to E$.

A map $\Phi : P \times G \to \underline{P}$ defining the action of G on $\underline{P}$; if $a, b \in G$ and $\varepsilon \in G$ is the unit element, then $\Phi(a) \circ \Phi(b) = \Phi(ba)$ and $\Phi(\varepsilon) = \mathrm{id}$ and $\Phi(a)p = \Phi(p, a) = R_a p = pa$; moreover $\pi \circ \Phi(a) = \pi$. ω is a 1-form of a connection on $\underline{P}$ with values in the Lie algebra of the group G. Let $\Phi'(a)$ be the tangent map to $\Phi(a)$, whereas $\Phi^*(a)$ is contragradient to $\Phi(a)$ at the point a. The form ω is a form of Ad-type, i.e.,

$$\Phi^*(a)\omega = \mathrm{Ad}_{a^{-1}}\omega \tag{1.1}$$

where $\mathrm{Ad}_a \in GL(g)$ is the tangent map to the internal automorphism of the group G (i.e., it is an adjoint representation of a group G)

$$\mathrm{ad}_a(b) = aba^{-1}$$

Due to the form ω, we can introduce the distribution field of linear elements H_r, $r \in \underline{P}$, where $H_r \subset T_r(\underline{P})$ is a subspace of the space tangent to $\underline{P}$ at a point r and

$$v \in H_r \Leftrightarrow \omega(v) = 0 \tag{1.2}$$

We have

$$T_r(\underline{P}) = V_r \oplus H_r \tag{1.3}$$

where H_r is called a subspace of horizontal vectors and V_r of vertical vectors. For vertical vectors $v \in V_r$ we have $\pi'(v) = 0$. This means that v is tangent to fibers. Let us define

$$v = \mathrm{hor}(v) + \mathrm{ver}(v), \qquad \mathrm{hor}(v) \in H_r, \qquad \mathrm{ver}(v) \in V_r \tag{1.4}$$

It is well known that the distribution H_r is equivalent to a choice of the connection ω. We can reproduce the connection form ω demanding that $\pi'|_{H_r}: H_r \to T_{\pi(r)}(E)$ is a vector space isomorphism ($\dim H_r = \dim E = 4$), $H_{\Phi(r,g)} = \Phi'(g) H_r$, [$T_{\pi(r)}(E)$ is a tangent space to space-time E at a point $\pi(r)$]. We use the operation "hor" for forms, i.e.,

$$(\mathrm{hor}\,\beta)(X, Y) = \beta(\mathrm{hor}\, X, \mathrm{hor}\, Y) \tag{1.5}$$

where X, $Y \in T_r(\underline{P})$. The 2-form of curvature of the connection ω is

$$\Omega = \mathrm{hor}\, d\omega \tag{1.6}$$

It is also a form of Ad-type like ω. Ω obeys the structural Cartan equation

$$\Omega = d\omega + \tfrac{1}{2}[\omega, \omega] \tag{1.7}$$

where

$$[\omega, \omega](X, Y) = [\omega(X), \omega(Y)]$$

Bianchi's identity for ω is

$$\mathrm{hor}\, d\Omega = 0 \tag{1.8}$$

For the principal fiber bundle we use the following convenient scheme (Figure 1A). The map $e: U \to \underline{P}$, $U \subset E$ (U open), so that $e \circ \pi = \mathrm{id}_U$ is called a local section. From the physical point of view it means choosing the gauge. Thus,

$$\begin{aligned} e^*\omega &= e^*(\omega^a X_a) = A^a{}_\mu \bar\theta^\mu X_a = A \\ e^*\Omega &= e^*(\Omega^a X_a) = \tfrac{1}{2} F^a_{\mu\nu} \bar\theta^\mu \wedge \bar\theta^\nu X_a \end{aligned} \tag{1.9}$$

Further, we introduce the notation

$$\Omega^a = \tfrac{1}{2} H^a_{\mu\nu} \theta^\mu \wedge \theta^\nu \tag{1.10}$$

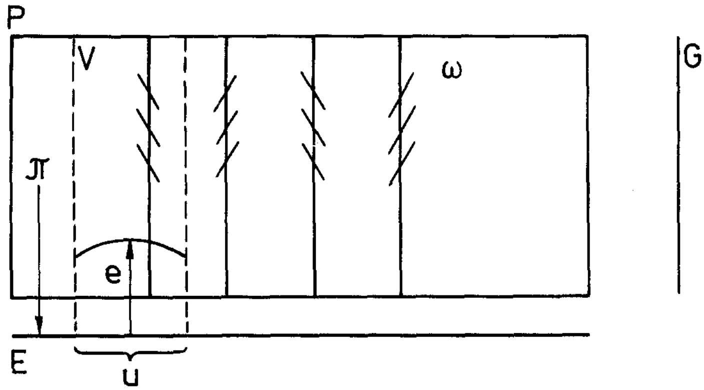

Fig. 1A. The principal fiber bundle $\underline{P}$.

where $\theta^{\mu} = \pi^{*}(\bar{\theta}^{\mu})$ and

$$F^{a}_{\mu\nu} = \partial_{\mu}A^{a}{}_{\nu} - \partial_{\nu}A^{a}{}_{\mu} + C^{a}_{bc}A^{b}_{\mu}A^{c}_{\nu}$$

X_a ($a = 1, 2, \ldots \dim G = n$) are generators of the Lie algebra $\mathfrak{g}$ of the group G and

$$[X_a, X_b] = C^{c}{}_{ab}X_c$$

Analogously we can introduce a second local section $f: U \to P$, and corresponding to it $\bar{A} = f^{*}\omega$, $\bar{F} = f^{*}\Omega$. For every $x \in U \subset E$ there is an element $g(x) \in G$ such that $f(x) = e(x)g(x) = R_{g(x)}e(x) = \Phi(e(x), g(x))$. Due to equation (1.1) and an analogous formula for Ω, one gets $A = \bar{A}d_{g^{-1}}A + g^{-1}\, dg$ and $\bar{F} = Ad_{g^{-1}}F$. These formulas give the geometrical meaning of gauge transformation.

In this paper we use also a linear connection on manifolds $\underline{P}$ and E using the formalism of differential forms. So the basic quantity is a 1-form of a connection $\omega^{A}{}_{B}$. This is an R-valued (coefficient) connection form and it is referred to the principal fiber bundle of frames with $\underline{P}$ or E as a base. The 2-form of curvature is

$$\Omega^{A}{}_{B} = d\omega^{A}{}_{B} + \omega^{A}{}_{C} \wedge \omega^{C}{}_{B} \tag{1.11}$$

and the 2-form of torsion

$$\Theta^{A} = D\theta^{A} \tag{1.12}$$

where θ^{A} are basic forms, and D means the exterior covariant derivative with respect to $\omega^{A}{}_{B}$. The following relations define the interrelation between our symbols and generally used ones:

$$\begin{aligned} \omega^A{}_B &= \Gamma^A{}_{BC}\theta^C \\ \Theta^A &= \tfrac{1}{2}Q^A{}_{BC}\theta^B \wedge \theta^C \\ \Omega^A{}_B &= \tfrac{1}{2}R^A{}_{BCD}\theta^C \wedge \theta^D \end{aligned} \tag{1.13}$$

Where $\Gamma^A{}_{BC}$ are coefficients of the connection (they do not have to be symmetric in indices B and C), $R^A{}_{BCD}$ is a tensor of curvature, and $Q^A{}_{BC}$ is a tensor of torsion. Covariant exterior differentiation with respect to $\omega^A{}_B$ is given by the formula

$$\begin{aligned} D\Xi^A &= d\Xi^A + \omega^A{}_C \wedge \Xi^C \\ D\Sigma^A{}_B &= d\Sigma^A{}_B + \omega^A{}_C \wedge \Sigma^C{}_B - \omega^C{}_B \wedge \Sigma^A{}_C \end{aligned} \tag{1.14}$$

The forms of curvature $\Omega^A{}_B$ and torsion Θ^A obey Bianchi identities

$$\begin{aligned} D\Omega^A{}_B &= 0 \\ D\Theta^A &= \Omega^A{}_B \wedge \theta^B \end{aligned} \tag{1.15}$$

In this paper we use also Einstein's + and − differentiations for the nonsymmetric metric tensor g_{AB}:

$$Dg_{\underset{+}{A}\underset{-}{B}} = Dg_{AB} - g_{AD}Q^D{}_{BC}\theta^C \tag{1.16}$$

where D is the convariant exterior derivative with respect to $\omega^A{}_B$ and $Q^D{}_{BC}$ is the tensor of torsion for $\omega^A{}_B$. In a homolonomic system of coordinates we easily get

$$Dg_{\underset{+}{A}\underset{-}{B}} = g_{\underset{+}{A}\underset{-}{B};C}\theta^C = [g_{AB,C} - g_{DB}\Gamma^D{}_{AC} - g_{AD}\Gamma^D{}_{CB}]\theta^C \tag{1.17}$$

All quantities introduced in this section and their precise definitions can be found in refs. 51 and 59–61.

Finally, let us connect a general formalism of the principal fiber bundle with a formalism of a linear connection on E or P.

Let M be an m-dimensional pseudo-Riemannian manifold with metric g of arbitrary signature. Let $T(M)$ be the tangent bundle and $O(M, g)$ the principal fiber bundle of frames (orthonormal frames) over M. The structure group of $O(M, g)$ is the group $Gl(m, \mathbb{R})$ or the subgroup of $Gl(m, \mathbb{R})$, $O(m-p, p)$, which leaves the metric invariant. Let Π be the projection of $O(M, g)$ onto M. Let X be a tangent vector at a point x in $O(M, g)$. The canonical or soldering form $\tilde{\theta}$ is an R^m-valued form on $O(M, g)$ whose Ath component $\tilde{\theta}^A$ at x of X is the Ath component of $\Pi'(X)$ in the frame

x. The connection form $\tilde{\omega} = \omega^A{}_B X^B{}_A$ is a 1-form on $O(M, g)$ which takes its values in the Lie algebra $gl(m, \mathbb{R})$ of $Gl(m, \mathbb{R})$ or in $o(m-p, p)$ of $O(m-p, p)$ satisfies the structure equations

$$d\tilde{\omega} + \tfrac{1}{2}[\tilde{\omega}, \tilde{\omega}] = \tilde{\Omega} = \tilde{\mathrm{H}}\mathrm{or}\ d\tilde{\omega} \tag{1.18}$$

where $\tilde{\mathrm{H}}\mathrm{or}$ is understood in the sense of $\tilde{\omega}$ and $\tilde{\Omega} = \tilde{\Omega}^A{}_B X^B{}_A$ is a $gl(m, \mathbb{R})$-$(o(m-p, p))$-valued 2-form of the curvature. We can write equation (1.18) using R^{2m}-valued forms and commutation relations of the Lie algebra $gl(m, \mathbb{R})(o(m, m-p))$,

$$\tilde{\Omega}^A{}_B = d\tilde{\omega}^A{}_B + \tilde{\omega}^A{}_C \wedge \tilde{\omega}^C{}_B \tag{1.19}$$

Taking any local section of $O(M, g)e$, one can get forms of coefficients of the connection, torsion, curvature, and basic forms

$$\begin{aligned} e^*\tilde{\omega}^A{}_B &= \omega^A{}_B \\ e^*\tilde{\Omega}^A{}_B &= \Omega^A{}_B \\ e^*\tilde{\theta}^A &= \theta^A \\ e^*\tilde{\theta}^A &= \Theta^A \end{aligned} \tag{1.20}$$

The forms of the right-hand side of equations (1.20) are the forms defined in equations (1.11), (1.12), (1.13), (1.14), etc. We call this formalism a linear (affine, metric, Riemannian, Einstein) connection on *M*.

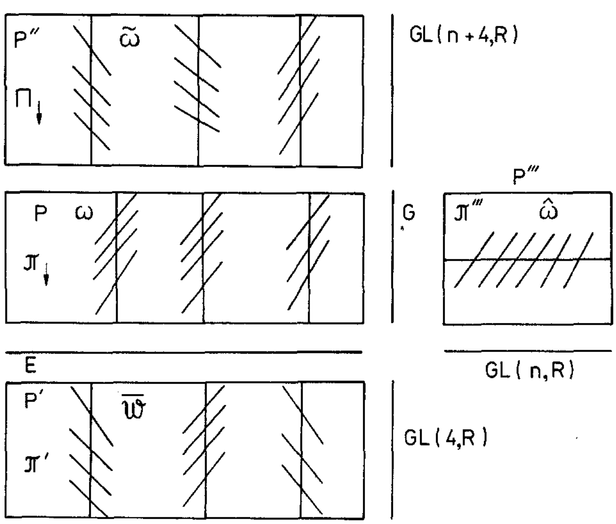

Fig. 1B. Principal fiber bundles P, P', P'', and P'''; P''' is a principal fiber bundle of frames over G.

In our theory it is necessary to consider at least four principal fiber bundles: a principal fiber bundle P over E with a structural group G (a gauge group), connection ω, and horizontality operator "hor"; a principal fiber bundle P' of frames over (E, g) with the connection $\tilde{\omega}^{\alpha}{}_{\beta}X^{\beta}{}_{\alpha} = \omega'$, a structural group $Gl(4, \mathbb{R})(O(1,3))$, and an operator of horizontality $\overline{\text{hor}}$; a principal fiber bundle P'' of frames over $(\underline{P}, \gamma)$ (a metrized fiber bundle $\underline{P}$) with a structural group $Gl(4+n, \mathbb{R})(O(n+3, 1))$, a connection $\tilde{\omega}^{A}{}_{B}X^{B}{}_{A} = \tilde{\omega}$, and an operator of horizontality $\overline{\text{hor}}$; and a principal fiber bundle of frames P''' over G with a projection Π''', operator of horizontality (hor)$'''$, a connection $\hat{\omega}$, and the structural group $Gl(n, \mathbb{R})$. Moreover, in order to simplify considerations, use the formalism of linear connection coefficients on manifolds (E, g), $(\underline{P}, \gamma)$ and a principal fiber bundle formalism for $\underline{P}$ (a principal fiber bundle over E with the structural group G a gauge group). This will make the formalism more natural and readable (see Figure 1B).

2. THE NONSYMMETRIC TENSOR ON A LIE GROUP

Let G be a Lie group and let us define on G a tensor field $h = h_{ab}v^{a} \otimes v^{b}$ and a field of a 2-form $k = k_{ab}v^{a} \wedge v^{b}$, where

$$dv^{a} = -\tfrac{1}{2}C^{a}{}_{bc}v^{b} \wedge v^{c} \tag{2.1}$$

v^{a} is a usual left-invariant frame on G, and $C^{a}{}_{bc}$ are structure constants. Let X_{a} be generators of a Lie algebra $G-\mathfrak{g}$; X_{a} are left-invariant vector fields on G and they are dual to the forms v^{a}

$$[X_{a}, X_{b}] = C^{c}{}_{ab}X_{c} \tag{2.2}$$

Using h and k, we construct a tensor field on G,

$$l_{ab} = h_{ab} + \mu k_{ab} \tag{2.3}$$

where μ is a real number. Let us recall that the left-invariant vector fields on G are infinitesimal transformations of a right action of G on G. The symbol $\mathrm{Ad}_{G}(g)$ means a matrix of the adjoint representation of the group G. We denote it Adg. R means a right action of the group G on G; L, a left action $[R(g), L(g), g \in G]$.

We are looking for the following h and k:

$$R^{*}(g)h = h \tag{2.4}$$

$$R^{*}(g)k = k \tag{2.5}$$

or in terms of the tensor l_{ab},

$$R^{*}(g)l = l \tag{2.6}$$

The condition (2.5) can be rewritten

$$(R^{*}(g))k_{g_1}(X_{g_1}, Y_{g_1}) = k_{g_1 g}(X_{g_1}g', Y_{g_1}g') = k_{g_1}(X_{g_1}, Y_{g_1}) \tag{2.5a}$$

where $g, g_1 \in G$. Moreover, X, Y are left-invariant vector fields on G. Thus, $X_g = X_\varepsilon = X$, $Y_g = Y_\varepsilon = Y$, and

$$(R^*(g))k_{g_1}(X, Y) = k_{g_1 g}(Xg', Yg') = k_{g_1}(X, Y) \tag{2.5b}$$

where $\varepsilon \in G$ is a unit element of G.

In order to find h and k satisfying (2.4) and (2.5), we define a linear connection on G such that

$$\tilde{\omega}^a{}_b = -C^a{}_{bc} v^c \tag{2.7}$$

Let the covariant differentiation with respect to $\hat{\omega}^a{}_b$ be $\hat{\nabla}_c$ and an exterior covariant differentiation $\hat{D}$. It is easy to see that this connection is flat,

$$\hat{\Omega}^a{}_b = d\hat{\omega}^a{}_b + \hat{\omega}^a{}_c \wedge \hat{\omega}^c{}_b = 0 \tag{2.8}$$

with nonzero torsion

$$\hat{\Theta}^a = \hat{D}v^a = dv^a + \hat{\omega}^a{}_b \wedge v^c = \tfrac{1}{2} C^a{}_{bc} v^b \wedge v^c \tag{2.8a}$$

and with a tensor of torsion

$$\hat{Q}^a{}_{bc} = C^a{}_{bc} \tag{2.9}$$

This connection is also metric. This means that the Killing-Cartan tensor on the group G is absolutely parallel with respect to $\hat{\omega}^a{}_b$. A parallel transport according to this connection is a right action of the group G on G.

One can easily find that (2.4)-(2.6) are equivalent to the condition

$$\hat{\nabla}_c l_{ab} = 0 \tag{2.10}$$

Thus, in order to find h and k, we should solve (2.10) on the group G. Let us prove that the system (2.10) is self-consistent.

In order to do this, let us consider the commutator of the covariant derivatives

$$2\hat{\nabla}_{[r}\hat{\nabla}_{k]} l_{cd} = \hat{R}^b{}_{crk} l_{bd} + \hat{R}^b{}_{drk} l_{cb} + \hat{Q}^p{}_{rk} \hat{\nabla}_p l_{cd} \tag{2.11}$$

Moreover, $\hat{\omega}^a{}_b$ is flat and we get

$$\begin{gathered} 2\hat{\nabla}_{[r}\hat{\nabla}_{k]} l_{cd} = \hat{Q}^p{}_{rk} \hat{\nabla}_p l_{cd} = C^p{}_{rk} \hat{\nabla}_p l_{cd} \\ \hat{\nabla}_p l_{cd} = 0 \end{gathered} \tag{2.12}$$

which proves the consistency of (2.10).

This result we can get using the equivalent form of (2.10),

$$X_f l_{cd} + l_{nd} C^n_{cf} + l_{cn} C^n_{df} = 0 \tag{2.13}$$

It is easy to see that a bi-invariant tensor h on G satisfies (2.13) identically (for example, a Killing–Cartan tensor).

Thus, one gets for a tensor k_{ab}

$$\hat{\nabla}_c k_{ab} = X_c k_{ab} + k_{nb} C^n_{ac} + k_{an} C^n_{bc} = 0 \tag{2.14}$$

It is easy to see that if k_{ab} satisfies (2.14), $b \cdot k_{ab}$ satisfies this condition as well for $b = \text{const.}$

In the case of an Abelian group, k is bi-invariant on G.

The interesting case in our theory is a semisimple group G. In this case k_{ab} cannot be bi-invariant. The only bi-invariant 2-form on the semisimple Lie group G is a zero form. Moreover, equation (2.14) has always a solution on a semisimple group and k is right-invariant. Moreover, we suppose that the symmetric part of l is bi-invariant (left- and right-invariant) and k only right-invariant.

We can also define k in a special way,

$$k(A, B) = h([A, B], V), \qquad A = A^a X_a, \quad B = B^a X_a \tag{2.15}$$

where

$$\hat{\nabla}_c V_d = 0 \tag{2.16}$$

$V = V_d \otimes v^d$ is a covector field on G (it is right-invariant) and h is a Killing–Cartan tensor on G.

In order to become more familiar with the notion of a tensor k, we find it for the group $SO(3)$. In this case we have left-invariant vector fields

$$\begin{aligned} e_1 &= \cos\psi \frac{\partial}{\partial\theta} - \sin\psi\left(\cot\theta \frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) \\ e_2 &= \sin\psi \frac{\partial}{\partial\theta} + \cos\psi\left(\cot\theta \frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) \\ e_3 &= \frac{\partial}{\partial\psi} \end{aligned} \tag{2.17}$$

such that

$$[e_a, e_b] = -\varepsilon_{abc} e_c; \qquad a, b, c = 1, 2, 3 \tag{2.18}$$

θ, ϕ, ψ are Euler angles—the usual parametrization of $SO(3)$,

$$\begin{aligned} 0 &\le \theta \le \pi \\ 0 &\le \psi \le 2\pi \\ 0 &\le \phi \le 2\pi \end{aligned} \tag{2.19}$$

and $\varepsilon_{129}=1$ and ε_{abc} is a Levi-Civita symbol (see ref. 62). In this case one can easily integrate (2.16) and find

$$\begin{aligned}
V_1(\theta,\phi,\psi) &= a(\cos\phi\cos\psi-\cos\theta\sin\phi\sin\psi)\\
&\qquad +b\sin\psi\sin\theta-c(\sin\phi\cos\psi+\cos\theta\cos\phi\sin\psi)\\
V_2(\theta,\phi,\psi) &= a(\sin\psi\cos\phi+\cos\theta\cos\psi\sin\phi)\\
&\qquad -b\cos\psi\sin\theta+c(\cos\theta\cos\phi\cos\psi-\sin\phi\sin\psi)\\
V_3(\theta,\phi,\psi) &= a\sin\phi\sin\theta+b\cos\theta+c\sin\phi\sin\theta, \qquad a,b,c=\text{const}
\end{aligned} \tag{2.20}$$

In the simpler case $a=c=0$, $b\neq 0$, one gets

$$\begin{aligned}
V_1 &= b\sin\theta\sin\psi\\
V_2 &= -b\sin\theta\cos\psi\\
V_3 &= b\cos\theta, \qquad b=\text{const}
\end{aligned} \tag{2.20a}$$

For

$$k_{ab}=\varepsilon_{abc}V_c \tag{2.21}$$

we get

$$k_{ab}=\begin{pmatrix}
0 & \begin{matrix}(a\sin\phi\sin\theta+b\cos\theta\\ +c\sin\phi\sin\theta)\end{matrix} & \begin{matrix}-[a(\sin\psi\cos\phi\\ +\cos\theta\cos\psi\sin\phi)\\ -b\cos\psi\sin\theta\\ +c(\cos\theta\cos\phi\cos\psi\\ -\sin\phi\sin\psi)]\end{matrix}\\
\begin{matrix}-(a\sin\phi\sin\theta+b\cos\theta\\ +c\sin\phi\sin\theta)\end{matrix} & 0 & \begin{matrix}[a(\cos\phi\cos\psi\\ -\cos\theta\sin\phi\sin\psi)\\ +b\sin\psi\sin\theta\\ -c(\sin\phi\cos\psi\\ +\cos\theta\cos\phi\sin\psi)]\end{matrix}\\
\begin{matrix}a(\sin\psi\cos\phi\\ +\cos\theta\cos\psi\sin\phi)\\ -b\cos\psi\sin\theta\\ +c(\cos\theta\cos\phi\cos\psi\\ -\sin\phi\sin\psi)\end{matrix} & \begin{matrix}-[a(\cos\phi\cos\psi\\ -\cos\theta\sin\phi\sin\psi)\\ +b\sin\psi\sin\theta\\ -c(\sin\phi\cos\psi\\ +\cos\theta\cos\phi\sin\psi)]\end{matrix} & 0
\end{pmatrix} \tag{2.22}$$

In a simpler case for $a=c=0$, $b\neq 0$ one gets

$$k_{ab}=\begin{pmatrix}
0, & b\cos\theta, & -b\sin\theta\cos\psi\\
-b\cos\theta, & 0, & b\sin\theta\cos\psi\\
b\sin\theta\cos\psi, & -b\sin\theta\sin\theta\sin\psi, & 0
\end{pmatrix} \tag{2.22a}$$

Thus, if we choose for h a Killing–Cartan tensor on $SO(3)$ [this is a unique bi-invariant tensor on $SO(3)$ *modulo* constant factor]

$$h_{ab} = -2\delta_{ab} \tag{2.23}$$

we easily get

$$l_{ab} = \begin{pmatrix} -2 & \begin{array}{l}\mu(\alpha \sin\phi \sin\theta \\ \quad +\beta\cos\theta \\ \quad +\gamma \sin\phi \sin\theta)\end{array} & \begin{array}{l}-\mu[\alpha(\sin\psi\cos\phi \\ \quad +\cos\theta\cos\psi\sin\phi) \\ \quad -\beta\cos\psi\sin\theta \\ \quad +\gamma(\cos\theta\cos\phi\cos\psi \\ \quad -\sin\phi\sin\psi)]\end{array} \\ & -2 & \begin{array}{l}\mu[\alpha(\cos\phi\cos\psi \\ \quad -\cos\theta\sin\phi\sin\psi) \\ \quad +\beta\sin\psi\sin\theta \\ \quad -\gamma(\sin\phi\cos\psi \\ \quad +\cos\theta\cos\phi\sin\psi)]\end{array} \\ \begin{array}{l}\mu[\alpha(\sin\psi\cos\phi \\ \quad +\cos\theta\cos\psi\sin\phi) \\ \quad -\beta\cos\psi\sin\theta \\ \quad +\gamma(\cos\theta\cos\phi\cos\psi \\ -\sin\phi\sin\psi)]\end{array} & \begin{array}{l}-\mu[\alpha(\cos\phi\cos\psi \\ \quad -\cos\theta\sin\phi\sin\psi) \\ \quad +\beta\sin\psi\sin\theta \\ \quad -\gamma(\sin\phi\cos\psi \\ \quad +\cos\theta\cos\phi\sin\psi)]\end{array} & -2 \end{pmatrix} \tag{2.24}$$

where $\mu = \eta(a^2+b^2+c^2)^{1/2}$, $\eta^2 = 1$, $\alpha = a/\mu$, $\beta = b/\mu$, $\gamma = c/\mu$. In a simpler case, for $a = c = 0$, $b \neq 0$, one gets (absorbing β by μ):

$$l_{ab} = \begin{pmatrix} -2, & \mu\cos\theta, & \mu\sin\theta\cos\psi \\ -\mu\cos\theta, & -2, & \mu\sin\theta\cos\psi \\ -\mu\sin\theta\cos\psi, & -\mu\sin\theta\cos\psi, & -2 \end{pmatrix} \tag{2.24a}$$

For an inverse tensor l^{ab} such that

$$l^{ab}l_{ac} = l^{ba}l_{ca} = \delta^b{}_c \tag{2.25}$$

we have

$$l^{ab} = \frac{\Delta^{ab}}{\Delta} \tag{2.26}$$

where $\Delta = \det(l_{ab}) = -2(4+\mu^2)$, Δ^{ab} is a cofactor matrix, and

$$\begin{aligned}
\Delta^{11} &= 4+\mu^2 \sin^2\theta \sin^2\psi \\
\Delta^{12} &= -(2\mu \cos\theta + \mu^2 \sin^2\theta \sin\psi \cos\psi) \\
\Delta^{13} &= (\mu^2 \cos\theta \sin\theta \sin\psi - 2\mu \sin\theta \cos\psi) \\
\Delta^{21} &= (2\mu \cos\theta - \mu^2 \sin^2\theta \sin\psi \cos\psi) \\
\Delta^{22} &= (4+\mu^2 \sin^2\theta \cos^2\psi) \\
\Delta^{23} &= -(2\mu \sin\theta \sin\psi + \mu^2 \cos\theta \sin\theta \cos\psi) \\
\Delta^{31} &= (\mu^2 \cos\theta \sin\theta \sin\psi + 2\mu \sin\theta \cos\psi) \\
\Delta^{32} &= (2\mu \sin\theta \sin\psi - \mu^2 \cos\theta \sin\theta \cos\psi) \\
\Delta^{33} &= (4+\mu^2 \cos^2\theta)
\end{aligned} \tag{2.27}$$

In the case of $SO(3)$, equation (2.22) is the most general tensor satisfying (2.5) except for a constant factor in front. Thus, this tensor is unique for $SO(3)$ *modulo* a constant factor.

In the case of any $SO(n)$ one can find k and l similarly using Euler angle parametrization and so for classical groups $SU(n)$, $Sp(2n)$, G_2, F_4, E_6, E_7, E_8. In the case of solvable and nilpotent groups we can also try to find bi-invariant skew-symmetric tensors.

Finally, we suggest a general form of the tensor k_{ab} on a semisimple group G, i.e., such that equation (2.4) is satisfied. The solutions of equations (2.10) and (2.14) are as follows:

$$l_{ab}(e^C) = l_{a'b'}(\varepsilon)(e^{Ad'C})^{a'}_{a}(e^{Ad'C})^{b'}_{b}$$

and

$$k_{ab}(e^C) = k_{a'b'}(\varepsilon)(e^{Ad'C})^{a'}_{a}(e^{Ad'C})^{b}_{b}$$

One writes

$$k_{ab}(g) = f_{a'b'}U^{a'}{}_{a}(g)U^{b'}{}_{b}(g), \qquad g \in G \tag{2.28}$$

where $U(g) = \mathrm{Ad}_G(g)$ is an adjoint representation of the group G. It is easy to see that for (2.28) we have

$$\hat{\nabla}_c k_{ab} = 0 \tag{2.29}$$

$$f_{ab} = -f_{ab} = \text{const} \tag{2.30}$$

and it is defined in the representation space of the adjoint representation of the group G. In the case of the group $SO(3)$ one has

$$f_{ab} = \varepsilon_{abc} f_c \tag{2.31}$$

$$k_{ab} = \varepsilon_{abc} V_c \tag{2.31a}$$

and

$$V_a = f_{c'} U^{c'}{}_a(g) \tag{2.32}$$

If we choose $f_c = (0, 0, b)$, we get equation (2.20a). Moreover, it is always possible because an orthogonal $[SO(3)]$ transformation can transform any vector f into $(0, 0, \pm\|f\|)$, where $\|f\|$ is the length of f. The semisimple Lie group G can be considered a Riemannian manifold equipped with a bi-invariant tensor h (a Killing–Cartan tensor) and a connection induced by this tensor. This Riemannian manifold has a constant curvature. Such a manifold has a maximal group of isometries H of dimension $\frac{1}{2}n(n+1)$, $n = \dim G$ (see ref. 59) (the isometry is here understood in the sense of the metric measured along geodetic lines in Riemannian geometry induced by a Killing–Cartan tensor). This group is a Lie group. It is easy to see that for $G = SO(3)$ we have $H = SO(3) \otimes SO(3)$ and $\dim SO(3) \otimes SO(3) = 6$, $\dim SO(3) = 3$. The group $SO(3)$ leaves the Killing–Cartan tensor h_{ab} invariant,

$$h_{a'b'} A^{a'}{}_a A^{b'}{}_b = h_{ab} \tag{2.33}$$

where $A \in SO(3)$.

Moreover, f_{ab} has exactly three arbitrary parameters and solutions of equation (2.14) have the same freedom in arbitrary constants. This suggests that the tensor (2.28) could be in some sense unique *modulo* an isometry on $SO(3)$ and a constant factor b. In this case the classification of k_{ab} tensors on $SO(3)$ could be reduced to the classification of skew-symmetric tensors f_{ab} with respect to the action of the group $SO(3)$. In general the situation is more complex, because $SO(n)$, $n = \dim G$, does not leave the commutator (Lie bracket) invariant.

Let us suppose that G is compact. In this case we should find all inequivalent f_{ab} tensors with respect to an orthogonal transformation $A \in SO(n)$. It means we should transform f_{ab} to a canonical form via an orthogonal matrix, i.e.,

$$(f_{ab}) = f \to f' = (f'_{ab}) = A^T f A = A^{-1} f A \tag{2.34}$$

For skew-symmetric matrices we have the following canonical forms, the so-called block-diagonal matrices:

$$f = \begin{bmatrix} 0 & \xi^1 & & & \bigcirc \\ -\xi^1 & 0 & & & \\ & & \ddots & & \\ & \bigcirc & & 0 & \xi^m \\ & & & -\xi^m & 0 \end{bmatrix} \tag{2.35}$$

or for $n=2m+1$

$$f=\begin{bmatrix} 0 & \xi^1 & & & & \\ -\xi^1 & 0 & & & \bigcirc & \\ & & \ddots & & & \\ & & & 0 & \xi^m & 0 \\ & \bigcirc & & -\xi^m & 0 & 0 \\ & & & 0 & 0 & 0 \end{bmatrix} \tag{2.36}$$

where $\xi^1, \xi^2, \ldots, \xi^m$ are real numbers. In order to find them, we should solve a secular equation for f,

$$\det(\mu I_n - f) = \mu^n + a_1(f)\mu^{n-2} + a_2(f)\mu^{n-2} + \cdots$$
$$[I_n = (\delta^i_j)_{i,j=1,2\cdots n}] \tag{2.37}$$

The coefficients $a_1, a_2, \ldots$ are invariant with respect to an action of the group $O(n)$ $[SO(n)]$ and they are functions of $\xi^1, \ldots, \xi^m$. Thus, in the case of a compact semisimple Lie group, the skew-symmetric tensor k_{ab} on G is defined as

$$k_{ab}(g) = b \cdot \tilde{f}_{a'b'} U^{a'}{}_a(g) U^{b'}{}_b(g) \tag{2.38}$$

where b is a constant real factor and $(\tilde{f}_{ab}) = \tilde{f}$ is given by

$$\tilde{f} = A^T \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & \bigcirc & \\ & & 0 & \xi^1 & & & \\ & & -\xi^1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \xi^{m-1} \\ & & \bigcirc & & & -\xi^{m-1} & 0 \end{bmatrix} A \tag{2.39}$$

for $n=2m$, or

$$\tilde{f} = A^T \begin{bmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & \bigcirc & & \\ & & 0 & \xi^1 & & & & \\ & & -\xi^1 & 0 & \ddots & & & \\ & & & & & 0 & \xi^{m-1} & \\ & & \bigcirc & & & -\xi^{m-1} & 0 & \\ & & & & & & & 0 \end{bmatrix} A \tag{2.39a}$$

for $n=2m+1$.

Supposing that $h_{ab} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $n = 2m$ or $n = 2m+1$, one gets

$$l_{ab}(\varepsilon) = A^T \begin{bmatrix} \lambda_1 & \zeta^1 & & & & & \\ \zeta^1 & \lambda_2 & & & & \bigcirc & \\ & & \lambda_3 & \xi^2 & & & \\ & & -\xi^2 & \lambda_4 & & & \\ & & \bigcirc & & \ddots & & \\ & & & & & \lambda_{2m-1} & \xi^m \\ & & & & & -\zeta^m & \lambda_{2m} \end{bmatrix} A$$

$$= A^T \tilde{l}(\varepsilon) A \tag{2.39*}$$

for $n = 2m$ or

$$l_{ab}(\varepsilon) = A^T \begin{bmatrix} \lambda_1 & \zeta^1 & & & & & & \\ -\zeta^1 & \lambda_2 & & & & \bigcirc & & \\ & & \lambda_3 & \xi^2 & & & & \\ & & -\xi^2 & \lambda_4 & & & & \\ & & \bigcirc & & \ddots & & & \\ & & & & & \lambda_{2m-1} & \xi^m & \\ & & & & & -\zeta^m & \lambda_{2m} & \\ & & & & & & & \lambda_{2m+1} \end{bmatrix} A$$

$$= A^T \tilde{l}(\varepsilon) A \tag{2.39a*}$$

for $n = 2m+1$.

Moreover, if G is compact, we have $\lambda_i = \lambda$, $i = 1, 2, \ldots, n$, and $\lambda < 0$. This is because any bi-invariant symmetric tensor is proportional to the Killing–Cartan tensor. In particular, the Tr tensor commonly used in Yang–Mills theory is proportional to h_{ab}. Thus, $h_{ab} = \lambda(\mathrm{Tr})_{ab} = \lambda\delta_{ab}$, $\lambda < 0$. [For a particular normalization of generators $\mathrm{Tr}(\{X_a, X_b\}) = 2\delta_{ab}$.] Let us remark that, in general, if $k_{ab}(\varepsilon)$ and h_{ab} commute (for now I do not suppose that G is compact), we have $l_{ab}(\varepsilon) = (A^{-1}\tilde{l}(\varepsilon)A)_{ab}$, where $A \in Gl(n, \mathbb{R})$ and

$$l_{ab}(g) = U^{a'}{}_a(g) U^{b'}{}_b(g) (A^{-1}\tilde{l}(\varepsilon)A)_{a'b'}$$

One can say, of course, that $k_{ab'}$ tensors are defined with more arbitriness than bi-invariant, symmetric tensors. This is because k is only right-invariant.

Let us notice that

$$f_{ab} = k_{ab}(\varepsilon) \tag{2.40}$$

(ε is a unit element of G) and

$$R_{g'}k_{ab}(g) = k_{ab}(gg') = k_{cd}(g)U^c{}_a(g')U^d{}_b(g') \tag{2.41}$$

where $g, g' \in G$.

In the case of $G = SO(3)$, k_{ab} is unique up to an isometry of the Riemannian manifold with the bi-invariant tensor as a metric tensor and a constant factor. This suggests that the k_{ab} tensor given in the form (2.15)-(2.16) and (2.31)-(2.32) is an analogue of the Killing-Cartan tensor for k_{ab} (skew-symmetric). Moreover, the vector f can be transformed by an orthogonal $[O(n)]$ transformation into

$$\underset{n \text{ times}}{(0, 0, \ldots, \pm\|f\|)} \tag{2.42}$$

Thus, one gets

$$k_{ab}(g) = b \cdot C^c_{ab} f^b_{c'} U^{c'}_c(g) \tag{2.43}$$

where b is a constant factor and

$$(f^0_{c'}) = f^0 = \underbrace{(0, 0, \ldots, 1)}_{n \text{ times}} \tag{2.44}$$

Thus, we can write k in a more compact form

$$k(A, B)(g) = b \cdot h([A, B], \mathrm{Ad}_g f^0) \tag{2.45}$$

where $A = A^a X_a$, $B = B^a X_a$.

Using the bi-invariancy of the Killing-Cartan tensor, one can write

$$k(A, B)(g) = b \cdot h(\mathrm{Ad}_{g^{-1}}[A, B], f^0) \tag{2.45a}$$

Moreover, if there is $\tilde{g} \in G$ such that $\tilde{g}^2 = g$, we get

$$k(A, B)(\tilde{g}^2) = b \cdot h(\mathrm{Ad}_{\tilde{g}^{-1}}[A, B], \mathrm{Ad}_{\tilde{g}} f^0) \tag{2.46}$$

We find the interpretation of the factor b for K given by formulas (2.45)-(2.46). One gets

$$k_{ab}k^{ab} = h^{aa'}h^{bb'}k_{ab}k_{a'b'} = b^2\|\mathrm{Ad}_g f^0\|^2 = b^2 \tag{2.47}$$

Thus, we have

$$b = \pm(k_{ab}k^{ab})^{1/2} \tag{2.48}$$

Finally, let us note that we can repeat the considerations changing right (left)-invariant to left (right)-invariant everywhere. In this case we can consider a left-invariant 2-form k and a left-invariant nonsymmetric tensor on a Lie group G.

3. THE NONSYMMETRIC METRIZATION OF THE BUNDLE $\underline{P}$

Let us introduce the principal fiber bundle $\underline{P}$ over the space-time E with the structural group G and with the projection π. Let us suppose that (E, g) is a manifold with a nonsymmetric metric tensor of the signature $(-,-,-,+)$,

$$g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]} \tag{3.1}$$

Let us introduce a natural frame of $\underline{P}$,

$$\theta^A = (\pi^*(\bar{\theta}^\alpha), \theta^a = \lambda\omega^a), \qquad \lambda = \text{const} \tag{3.2}$$

It is convenient to introduce the following notations. Capital Latin indices A, B, C run over $1, 2, 3, \ldots, n+4$, $n = \dim G$. Lower case Greek indices are α, β, γ, $\delta = 1, 2, 3, 4$ and lower case Latin indices are a, b, c, $d = 5, 6, \ldots, n+4$. The bar atop θ^α and over other quantities indicates that these quantities are defined on E.

It is easy to see that the existence of the nonsymmetric metric on E is equivalent to introducing two independent geometrical quantities on E,

$$\bar{g} = g_{\alpha\beta}\bar{\theta}^\alpha \otimes \bar{\theta}^\beta = g_{(\alpha\beta)}\bar{\theta}^\alpha \otimes \bar{\theta}^\beta \tag{3.3}$$

$$\underline{g} = g_{\alpha\beta}\bar{\theta}^\alpha \wedge \bar{\theta}^\beta = g_{[\alpha\beta]}\bar{\theta}^\alpha \wedge \bar{\theta}^\beta \tag{3.4}$$

i.e., the symmetric metric tensor $\bar{g}$ on E and 2-form $\underline{g}$. On the group G we can introduce a bi-invariant symmetric tensor called the Killing–Cartan tensor,

$$h(A, B) = \mathrm{Tr}(\mathrm{Ad}'_A \circ \mathrm{Ad}'_B) \tag{3.5}$$

where $\mathrm{Ad}'_A(C) = [A, C]$ (it is tangent to Ad, i.e., it is an "infinitesimal" Ad transformation). It is easy to see that

$$h(A, B) = h_{ab}A^a \cdot B^b \tag{3.6}$$

where

$$h_{ab} = C^c{}_{ad}C^d{}_{bc}, \qquad h_{ab} = h_{ba}, \qquad A = A^a X_a, \qquad B = B^a X_a$$

This tensor is distinguished by the group structure, but there are of course other bi-invariant tensors on G. Normally it is supposed that G is semi-simple. It means that $\det(h_{ab}) \neq 0$. In this construction we use $l_{(ab)} = h_{ab}$ (the bi-invariant tensor on G) in order to get a proper limit (i.e., the non-Abelian Kaluza–Klein theory) for $\mu = 0$.

For a natural 2-form k on G, or a natural skew-symmetric right-invariant tensor, we choose k described in Section 2; k is zero for $U(1)$. Let us turn to the nonsymmetric natural metrization of $\underline{P}$. Let us suppose that

$$\bar{\gamma}(X, Y) = \bar{g}(\pi' X, \pi' Y) + \lambda^2\rho^2 h(\omega(X), \omega(Y)) \tag{3.7}$$

$$\underline{\gamma}(X, Y) = \underline{g}(\pi' X, \pi' Y) + \mu\lambda^2\rho^2 k(\omega(X), \omega(Y)) \tag{3.8}$$

μ = const and is dimensionless, X, $Y \in \operatorname{Tan}(\underline{P})$, and $\rho = \rho(x)$ is a scalar field on E. The first formula (2.9) was introduced by Trautman (in the case with $\rho = 1$) for the symmetric natural metrization of $\underline{P}$ and it was used to construct the Kaluza-Klein theory for $U(1)$ and non-Abelian generalizations of this theory. It is easy to see that

$$\bar{\gamma} = \pi^* \bar{g} \oplus \rho^2 h_{ab} \theta^a \otimes \theta^b \tag{3.9}$$

$$\underline{\gamma} = \pi^* \underline{g} + \mu \rho^2 k_{ab} \theta^a \wedge \theta^b \tag{3.10}$$

or

$$\gamma_{(AB)} = \left(\begin{array}{c|c} g_{[\alpha\beta]} & 0 \\ \hline 0 & \rho^2 h_{ab} \end{array}\right) \tag{3.11}$$

$$\gamma_{[AB]} = \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & \mu\rho^2 k_{ab} \end{array}\right) \tag{3.12}$$

For

$$\gamma_{AB} = \gamma_{(AB)} + \gamma_{[AB]}$$

one easily gets

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & \rho^2 l_{ab} \end{array}\right) \tag{3.13}$$

where $l_{ab} = h_{ab} + \mu k_{ab}$. Tensor γ_{AB} has this simple form in the natural frame on $\underline{P}$, θ^A. This frame is unholonomical, because

$$d\theta^a = \frac{\lambda}{2}\left(H^a{}_{\mu\nu} \theta^\mu \wedge \theta^\nu - \frac{1}{\lambda^2} C^a{}_{bc} \theta^b \wedge \theta^c \right) \neq 0 \tag{3.14}$$

γ is invariant with respect to the right action of the group on $\underline{P}$. In the case with $k_{ab} = 0$ we have

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & \rho^2 h_{ab} \end{array}\right) \tag{3.15}$$

For the electromagnetic case $[G = U(1)]$ one easily finds

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & -\rho^2 \end{array}\right) \tag{3.16}$$

Now let us take a section $e: E \to \underline{P}$ and attach to it a frame v^a, $a = 5, 6, \ldots, n+4$, selecting X^μ = const on a fiber in such a way that e is given by the condition $e^* v^a = 0$ and the fundamental fields ζ_a such that $v^a(\zeta_b) = \delta^a_b$ satisfy $[\zeta_b, \zeta_a] = (1/\lambda) C^c{}_{ab} \zeta_c$. Thus, we have

$$\omega = \frac{1}{\lambda} v^a X_a + \pi^*(A^a{}_\mu \bar{\theta}^\mu) X_a$$

where

$$e^*\omega = A = A^a{}_\mu \bar{\theta}^\mu X_a$$

In this frame the tensor γ takes the form

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} + \lambda^2\rho^2 l_{ab} A^a_\alpha A^b_\beta & \lambda\rho^2 l_{cb} A^c_\alpha \\ \hline \lambda\rho^2 l_{ac} A^c{}_\beta & \rho^2 l_{ab} \end{array}\right) \tag{3.17}$$

where

$$l_{ab} = h_{ab} + \mu k_{ab}$$

This frame is also unholonomic. One easily finds

$$dv^a = \frac{-1}{2\lambda} C^a{}_{bc} v^b \wedge v^c \tag{3.18}$$

The nonsymmetric theory of gravitation uses the nonsymmetric metric $g_{\mu\nu}$ such that

$$g_{\mu\nu} g^{\beta\nu} = g_{\nu\mu} g^{\nu\beta} = \delta^\beta_\mu \tag{3.19}$$

where the order of indices is important. If G is semisimple and $k_{ab} = 0$,

$$l_{ab} = h_{ab}, \qquad \det(h_{ab}) \neq 0$$

and

$$h_{ab} h^{bc} = \delta^c_a \tag{3.20}$$

Thus, one easily finds in this case

$$\gamma_{AC} \gamma^{BC} = \gamma_{CA} \gamma^{CB} = \delta^B_A \tag{3.21}$$

where the order of indices is important. We have the same for the electromagnetic case $[G = U(1)]$. In general, if $\det(l_{ab}) \neq 0$, then

$$l_{ab} l^{ac} = l_{ba} l^{ca} = \delta^c_b \tag{3.22}$$

where the order of indices is important. From (3.22) we have (3.21) for the general nonsymmetric metric γ.

It is easy to see that

$$\begin{aligned} \Phi'(g)\bar{\gamma} &= \bar{\gamma} \\ \Phi'(g)\underline{\gamma} &= \underline{\gamma} \end{aligned} \tag{3.23}$$

and γ_{AB} is an invariant tensor with respect to the right-action of the group G on $\underline{P}$.

In the case of any Abelian group the condition (3.23) is stronger and we get that γ_{AB} is bi-invariant. Thus, in the case of $G = U(1)$ (electromagnetic case)

$$\xi_5 \bar{\gamma} = 0 = \xi_5 \underline{\gamma} \tag{3.24}$$

where ξ_A is a dual base

$$\theta^A(\xi_B) = \delta^A_B \tag{3.25}$$

A, $B = 1, 2, 3, 4, 5$, and

$$\xi_A = (\xi_\alpha, \xi_5) \tag{3.26}$$

Let us come back to the connection $\hat{\omega}^a{}_b$ defined on the group G. For a typical fiber diffeomorphic to G, we can define $\hat{\omega}^a{}_b$ on every fiber $F_x \simeq G$, $x \in E$. Due to a local trivialization of the bundle P, we can define $\hat{\omega}^a{}_b$ on every set $U \times G$, where $U \subset E$ and is open. Thus, we get a linear connection on P such that

$$\hat{\omega}^A{}_B = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & -(1/\lambda) C^a{}_{bc} \theta^c \end{array}\right) \tag{3.27}$$

defined in a frame $\theta^A = (\pi^*(\bar{\theta}^\alpha), \theta^a)$, where $\bar{\theta}^\alpha$ is a frame on E and θ^a is a horizontal lift base.

This connection can be examined in a systematic way. Let us introduce a metric on P in the following way:

$$p = \pi^* \eta \oplus h_{ab} \theta^a \otimes \theta^b \tag{3.28}$$

where $\eta = \eta_{\mu\nu} \bar{\theta}^\mu \otimes \bar{\theta}^\nu$ is a Minkowski tensor and h_{ab} is a Killing-Cartan tensor on G. One gets

$$p_{AB} = \left(\begin{array}{c|c} \eta_{\alpha\beta} & 0 \\ \hline 0 & h_{ab} \end{array}\right) \quad \text{and} \quad p^{AB} = \left(\begin{array}{c|c} \eta^{\alpha\beta} & 0 \\ \hline 0 & h^{ab} \end{array}\right) \tag{3.29}$$

The connection $\hat{\omega}^A{}_B$ can be defined as

$$\hat{\omega}^A{}_B = \left(\begin{array}{c|c} \pi^* \hat{\bar{\omega}}^\alpha{}_\beta & 0 \\ \hline 0 & \phi^*_x \hat{\omega}^a{}_b \end{array}\right) \tag{3.27a}$$

where $\hat{\bar{\omega}}^\alpha{}_\beta$ is a trivial connection on the Minkowski space, $\hat{\omega}^a{}_b$ is the connection defined in Section 2, and ϕ_x is a diffeomorphism $\phi_x : F_x \to G$, $x \in U$.

It is easy to check that

$$\hat{D} p_{AB} = 0 = \hat{D} p^{AB} \tag{3.30}$$

where $\hat{D}$ is an exterior covariant differential with respect to $\hat{\omega}^A{}_B$. One can easily calculate the torsion for $\hat{\omega}^A{}_B$,

$$\hat{Q}^a{}_{\mu\nu} = \lambda H^a{}_{\mu\nu} \tag{3.31a}$$

$$\hat{Q}^a{}_{bc} = \frac{1}{\lambda} C^a{}_{bc} \tag{3.31b}$$

and the curvature tensor

$$\hat{R}^a{}_{b\mu\nu} = \lambda X_b H^a{}_{\mu\nu} \tag{3.32}$$

(the remaining torsion and curvature components are zero). The connection $\hat{\omega}^A{}_B$ is neither flat nor torsionless. Moreover, it is still metric as a connection $\hat{\omega}^a{}_b$ from Section 2.

The covariant differentiation with respect to this connection is connected to the right action of the group G on $\underline{P}$. Thus, the condition of the right-invariance of the p-form $\Xi^{A_1\cdots A_l}{}_{B_1\cdots B_m}$ on $\underline{P}$ is equivalent to

$$\hat{\nabla}_a \Xi^{A_1\cdots A_l}{}_{B_1\cdots B_m} = 0 \tag{3.33}$$

where $\hat{\nabla}_k$ is a covariant derivative with respect to $\hat{\omega}^A{}_B$ in vertical directions on $\underline{P}$. This means right-invariance of Ξ. This can be written

$$\hat{\nabla}_{\mathrm{ver}(x)} \Xi = 0 \tag{3.33a}$$

ver is understood in the sense of ω.

$$\Phi'(g)\Xi = \Xi \tag{3.34}$$

where $g \in G$ and

$$\Xi = (\Xi^{A_1\cdots A_l B_1\cdots B_m}) = (p^{b_1 B_1'} p^{B_2 B_2'} \cdots p^{B_m B_m'} \Xi^{A_1\cdots A_l}{}_{B_1'\cdots B_m'})$$

For a connection ω on a bundle $\underline{P}$, with curvature Ω, one gets

$$\hat{\nabla}_k \omega = \hat{\nabla}_k \Omega = 0 \tag{3.34*}$$

Thus, we can rewrite equation (3.23),

$$\hat{\nabla}_a \underline{\gamma} = \hat{\nabla}_a \tilde{\gamma} = 0 \tag{3.35}$$

This means that

$$\hat{\nabla}_a \gamma_{AB} = 0 \tag{3.36}$$

or

$$\hat{\nabla}_{\mathrm{ver}(x)} \gamma = 0 \tag{3.36a}$$

For every linear connection $\omega^A{}_B$ defined on $\underline{P}$ compatible in some sense with γ_{AB} we get

$$\Phi^*(g)\omega_{AB} = \mathrm{Ad}_g \omega_{AB} \tag{3.37}$$

which means that ω_{AB} is right-invariant with respect to the right-action of the group G on $\underline{P}$. We say the same for a 2-form of torsion and 2-form of curvature derived for $\omega^A{}_B$, i.e.,

$$\hat{\nabla}_a\Omega^A{}_B = \hat{\nabla}_a\Theta^A = 0 \tag{3.38}$$

The curvature scalar is invariant with respect to the right-action of the group G on P,

$$0 = \hat{\nabla}_a R = X_a R = \zeta_a R \tag{3.39}$$

The condition (3.37) is the same as in the classical Kaluza–Klein (Jordan–Thiry) theory in a non-Abelian case. A parallel transport with respect to the connection $\hat{\omega}^A{}_B$ means simply a right-action of the group G on P.

Our subject of investigation consists in looking for a generalization of the geometry from Einstein's unified field theory (the so-called Einstein-Kaufman theory; (see refs. 4, 5, and 61) defined on $\underline{P}$ i.e., for a connection $\omega^A{}_B$ such that

$$D\gamma_{AB} = \gamma_{AD}Q^D{}_{BE}\theta^E \tag{3.40}$$

where D is an exterior, covariant differential with respect to the connection $\omega^A{}_B$, and $Q^D{}_{BE}$ is a tensor of torsion for $\omega^A{}_B$. We suppose that this connection is right-invariant with respect the right-action of the group G.

We can write equations (3.37)–(3.39) for a torsion, curvature, and a scalar of curvature for $\omega^A{}_B$. In this way we consider an Einstein–Kaufman G-structure on the bundle of linear frames over the manifold $\underline{P}$ (i.e., a right G-structure).

We can repeat all the considerations changing right (left)-invariant into left (right)-invariant in all places.

In Appendix A we consider in more detail the invariance properties of the connection $\omega^A{}_B$ from a different point of view.

In this section we define $\omega^A{}_B$ as a collection of 1-forms defined on the manifold P (a gauge bundle manifold) and we choose for $\omega^A{}_B$ a lift horizontal frame (connected to the connection ω on the gauge bundle). The collection of 1-forms $\omega^A{}_B$ becomes a linear connection on $\underline{P}$ iff it satisfies the following transformation properties:

$$\omega'^{A'}{}_{B'} = \Sigma^{-1A'}{}_A(p)\omega^A{}_B\Sigma^{-1B}{}_{B'}(p) - \Sigma^{-1A'}{}_A(p)\, d\Sigma^A{}_{B'}(p) \tag{3.41}$$

where

$$\Sigma(p) \in GL(n+4, \mathbb{R}), \qquad p \in U_p \subset \underline{P}$$

and

$$\theta^c = \Sigma^c{}_{c'}(p)\theta'^{c'} \tag{3.42}$$

is a simultaneous transformation property of a frame. Having $\omega^A{}_B$ with transformation properties (3.41)-(3.42), we can lift it on a principal fiber bundle of frames over $\underline{P}$ with the structural group $GL(n+4,\mathbb{R})$, getting a 1-form of connection $\tilde{\omega}$,

$$\tilde{\omega}_z = \mathrm{Ad}_{Gl(n+4,R)}(g_p^{-1})[\Pi^*(\omega^A{}_B X^B{}_A) - g_p^{-1}\, dg_p] \tag{3.43}$$

where Π is a projection defined on this principal fiber bundle of frames and

$$g_p: \quad z \in \Pi^{-1}(U_p) \to g_p(z) = (\mathrm{pr}_{GL(n+4,R)}\Psi_p(z))^{-1} \in Gl(n+4,\mathbb{R}), \quad p \in U_p \subset \underline{P}$$

pr means a projection on $Gl(n+4,\mathbb{R})$ in a local trivialization of the bundle P'', Ψ is an action of $GL(n+4,\mathbb{R})$ on a principal fiber bundle of frames over P, $\Psi \to Gl(n+4,\mathbb{R}) \times P'' \to P''$, and Ψ_p is defined for $Gl(n+4,\mathbb{R}) \times U_p$. In this way we have an action of $GL(n+4,\mathbb{R})$ on the bundle and for $\tilde{\omega}$,

$$\Psi^*(g)\tilde{\omega} = \mathrm{Ad}_{GL(n+4,R)}[g^{-1}]\tilde{\omega} \tag{3.44}$$

$X^A{}_B$ are generators of the Lie algebra $gl(n+4,\mathbb{R})$ of $GL(n+4,\mathbb{R})$ and $g \in Gl(n+4,\mathbb{R})$.

For a soldering form $\tilde{\theta}^A$ one gets

$$\tilde{\theta}^A = g_p \Pi^*(\theta^A) \tag{3.45}$$

Taking any two sections of the principal fiber bundle of P'' frames E and F such that

$$\begin{aligned} E^*\tilde{\omega} &= \omega'^{A'}{}_{B'} X^{B'}{}_{A'} \\ F^*\tilde{\omega} &= \omega^A{}_B X^B{}_A \end{aligned} \tag{3.46}$$

$$\begin{aligned} E^*\tilde{\theta}^A &= \theta'^A \\ F^*\tilde{\theta}^A &= \theta^A \end{aligned} \tag{3.47}$$

one gets the transformation properties (3.41) and (3.42). In such a way that

$$E(p) = F(p)\Sigma(p) \tag{3.48}$$

equation (3.40) can be rewritten in a more compact form

$$\nabla\gamma = S \tag{3.49}$$

where

$$S(X,Y,Z) = [\mathrm{Tr}(\gamma \otimes Q)](X,Y,Z) = \sum_A \gamma(X,e_A)\theta^A(Q(Y,Z))$$

$$Q(Y,Z) = -Q(Z,Y)$$

is a torsion of the connection $\tilde{\omega}$; X, Y, Z are contravariant vector fields; and θ^A, e_B, $\theta^A(e_B) = \delta^A{}_B$, are dual bases.

Or, in a different form,

$$\nabla_Z\gamma(X,Y) = S(X,Y,Z) \tag{3.50}$$

∇ is a covariant derivative with respect to the connection $\tilde{\omega}$ on the fiber bundle of frames.

Moreover, now we consider γ, Q, X, Y, Z, etc., as geometrical objects living on appropriate associated fiber bundles to the fiber bundle of frames. The condition (3.50) gives us the Einstein–Kaufman connection $\tilde{\omega}$ on the principal fiber bundle of frames over P. For $\tilde{\omega}$ right-invariant with respect to the action of group G on this bundle of frames (lifted to this bundle from $\underline{P}$; see Appendix A for more details) the condition (3.50) is also right-invariant.

4. FORMULATION OF THE NONSYMMETRIC JORDAN–THIRY THEORY

Let P be the principal fiber bundle with the structural group G, over space-time E with a projection π, and let us define on this bundle a connection ω. Let us suppose that G is semisimple and that its Lie algebra is $\mathfrak{g}$. On space-time E we define a nonsymmetric metric tensor such that

$$\begin{gathered} g_{\alpha\beta} = g_{(\alpha\beta)} + g_{[\alpha\beta]} \\ g_{\alpha\beta}g^{\gamma\beta} = g_{\beta\alpha}g^{\beta\gamma} = \delta^{\gamma}{}_{\alpha} \end{gathered} \tag{4.1}$$

where the order of indices is important. We define also on E two connections $\bar{\omega}^{\alpha}{}_{\beta}$ and $\bar{W}^{\alpha}{}_{\beta}$,

$$\bar{\omega}^{\alpha}{}_{\beta} = \bar{\Gamma}^{\alpha}{}_{\beta\gamma}\bar{\theta}^{\gamma} \tag{4.2}$$

and

$$\begin{gathered} \bar{W}^{\alpha}{}_{\beta} = \bar{W}^{\alpha}{}_{\beta\gamma}\bar{\theta}^{\gamma} \\ \bar{W}^{\alpha}{}_{\beta} = \bar{\omega}^{\alpha}{}_{\beta} - \tfrac{2}{3}\delta^{\alpha}{}_{\beta}\bar{W} \end{gathered} \tag{4.3}$$

where

$$\bar{W} = \bar{W}_{\gamma}\bar{\theta}^{\gamma} = \tfrac{1}{2}(\bar{W}^{\sigma}{}_{\gamma\sigma} - \bar{W}^{\sigma}{}_{\sigma\gamma})\bar{\theta}^{\gamma}$$

For the connection $\bar{\omega}^{\alpha}{}_{\beta}$ we suppose the following conditions:

$$\begin{gathered} \bar{D}g_{\alpha+\beta-} = \bar{D}g_{\alpha\beta} - g_{\alpha\delta}\bar{Q}^{\delta}{}_{\beta\gamma}(\bar{\Gamma})\bar{\theta}^{\gamma} = 0 \\ \bar{Q}^{\alpha}{}_{\beta\alpha}(\bar{\Gamma}) = 0 \end{gathered} \tag{4.4}$$

where $\bar{D}$ is the exterior covariant derivative with respect to $\bar{\omega}^{\alpha}{}_{\beta}$ and $\bar{Q}^{\alpha}{}_{\beta\gamma}(\bar{\Gamma})$ is the torsion of $\bar{\omega}^{\alpha}{}_{\beta}$. Now let us turn to the natural nonsymmetric metrization of the bundle $\underline{P}$. According to Section 3, we have

$$\begin{gathered} \bar{\gamma} = \pi^*\bar{g} \oplus \rho^2 h_{ab}\theta^a \qquad \theta^b = \pi^*(g_{(\alpha\beta)}\bar{\theta}^{\alpha} \otimes \bar{\theta}^{\beta}) \oplus \rho^2 h_{ab}\theta^a \otimes \theta^b \\ \underline{\gamma} = \pi^*\underline{g} \oplus \rho^2 \mu k_{ab}\theta^a \wedge \theta^b = \pi^*(g_{[\alpha\beta]}\bar{\theta}^{\alpha} \wedge \bar{\theta}^{\beta}) \oplus \rho^2 \mu k_{ab}\theta^a \wedge \theta^b \end{gathered} \tag{4.5}$$

where $\theta^a = \lambda\omega^a$. In Appendix A we consider a more general structure and give a justification for this special form. From the classical Kaluza-Klein theory and Jordan-Thiry theory (with symmetric metric) we know that $\lambda = 2$ (refs. 1, 16, 17, and 58). We have

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & \rho^2 l_{ab} \end{array}\right) \tag{4.6}$$

where

$$l_{ab} = h_{ab} + \mu k_{ab}$$

We suppose $l_{(ab)} = h_{ab}$ (the bi-invariant tensor on G) in order to get a proper limit for the Yang-Mills Lagrangian for $\mu = 0$. It is worth noticing that our results are valid for an arbitrary right-invariant nonsymmetric tensor l_{ab}.

Let us suppose that $\det(l_{ab}) \neq 0$. Now we define on P, a connection $\omega^A{}_B$ (right-invariant with respect to the right action of the group G on P) such that

$$D\gamma_{A+B-} = D\gamma_{AB} - \gamma_{AD}Q^D{}_{BC}(\Gamma)\theta^C = 0 \tag{4.7}$$

where

$$\omega^A{}_B = \Gamma^A{}_{BC}\theta^C$$

and

$$\Phi'(g)\omega_{AB} = \mathrm{Ad}_g\omega_{AB}, \qquad g \in G$$

D is the exterior covariant derivative with respect to the connection $\omega^A{}_B$ and $Q^D{}_{BC}(\Gamma)$ is the tensor of torsion for the connection $\omega^A{}_B$. Equation (4.7) means the compatibility condition in Einstein-Kaufman sense. $\Phi'(g)$ is a right-action of the group G on P. After some calculations one gets

$$\begin{aligned}
\Gamma^\alpha{}_{\beta\gamma} &= \tilde{\Gamma}^\alpha{}_{\beta\gamma} \\
\Gamma^d{}_{\beta\gamma} &= L^d{}_{\beta\gamma} \\
\Gamma^\beta{}_{\gamma b} &= -\rho^2 l_{db} g^{\alpha\beta} L^d{}_{\alpha\gamma} \\
\Gamma^\mu{}_{a\gamma} &= \rho^2 l_{ad} g^{\mu\beta}(2H^d{}_{\gamma\beta} - L^d{}_{\gamma\beta}) \\
\Gamma^\delta{}_{ac} &= N^\delta{}_{ac} \\
\Gamma^a{}_{\alpha c} &= -\frac{1}{\rho^2} g_{\alpha\delta} l^{ab} N^\delta{}_{cb} \\
\Gamma^b{}_{c\beta} &= -\frac{1}{\rho^2} g_{\delta\beta} l^{ab} N^\delta{}_{ac} \\
\Gamma^a{}_{bc} &= \tilde{\Gamma}^a{}_{bc}
\end{aligned} \tag{4.8}$$

where $L^{d}{}_{\beta\gamma}$, $N^{\rho}{}_{cb}$ are Ad-type tensors on P such that

$$(\rho^2)_{,\gamma} l_{ab} + (l_{db} l^{ed} g_{\delta\gamma} N^{\delta}{}_{ea} + l_{ad} l^{de} g_{\gamma\delta} N^{\delta}{}_{be}) = 0 \tag{4.9}$$

$$l_{dc} g_{\mu\beta} g^{\gamma\mu} L^{d}{}_{\gamma\alpha} + l_{cd} g_{\alpha\mu} g^{\mu\gamma} L^{d}{}_{\beta\gamma} = 2 l_{cd} g_{\alpha\mu} g^{\mu\gamma} H^{d}{}_{\beta\gamma} \tag{4.10}$$

and $\tilde{\Gamma}^{a}{}_{bc}$ satisfies the compatibility conditions:

$$l_{db} \tilde{\Gamma}^{d}{}_{ac} + l_{ad} \tilde{\Gamma}^{d}{}_{cb} = -l_{db} C^{d}{}_{ac} \tag{4.11}$$

The connection $\tilde{\omega}^{a}{}_{b} = \tilde{\Gamma}^{a}{}_{bc} \theta^{c}$ is defined on a typical fiber. According to our assumptions, $\tilde{\omega}^{a}{}_{b}$ is a right-invariant form on P. $\tilde{\Gamma}^{a}{}_{bc}$ in the lift horizontal basis has a tensorial transformation law (see Appendix A).

This means that

$$\Phi^*(g) \tilde{\omega}_{ab} = \mathrm{Ad}_g \tilde{\omega}_{ab} \tag{4.12}$$

or in coordinate language

$$X_k \tilde{\Gamma}^{d}{}_{ab} + C^{e}{}_{ak} \tilde{\Gamma}^{d}{}_{eb} + C^{e}{}_{bk} \tilde{\Gamma}^{d}{}_{ae} - C^{d}{}_{ek} \tilde{\Gamma}^{e}{}_{ab} = 0 \tag{4.13}$$

Using notations from Section 1, we can write equation (4.13) as

$$\hat{\nabla}_k \tilde{\Gamma}^{d}{}_{ab} = 0 \qquad \text{or} \qquad \hat{\nabla}_k \omega^{a}{}_{b} = 0 \tag{4.14}$$

and similarly

$$\hat{\nabla}_k l_{ab} = 0 \tag{4.15}$$

We can write for $\tilde{\Gamma}^{d}{}_{cb}$ the following:

$$R_{g'}(\tilde{\Gamma}^{d}{}_{cb}(g)) = \tilde{\Gamma}^{d}{}_{cb}(gg') = U^{d}{}_{d'}(g'^{-1}) U^{c'}{}_{c}(g') U^{b'}{}_{b}(g') \tilde{\Gamma}^{d'}{}_{c'b'}(g) \tag{4.16}$$

This allows us to write for $\tilde{\Gamma}^{d}{}_{cb}(g)$ the general formula

$$\tilde{\Gamma}^{d}{}_{cb}(g) = U^{d}{}_{d'}(g^{-1}) U^{c'}{}_{c}(g) U^{b'}{}_{b}(g) \Omega^{d'}{}_{c'b'} \tag{4.17}$$

where

$$\Omega^{d'}{}_{c'b'} = \tilde{\Gamma}^{d'}{}_{c'b'}(\varepsilon) \tag{4.18}$$

and is defined in the representation space of the adjoint representation of the group G. We now derive the following condition for $\Omega^{d'}{}_{c'b'}$:

$$(h_{db} + \mu f_{db}) \Omega^{d}{}_{ac} + (h_{ad} + \mu f_{ad}) \Omega^{d}{}_{cb} = -(h_{db} + \mu f_{db}) C^{d}{}_{ac} \tag{4.19}$$

For example, we can take $f_{ad} = C^{k_0}{}_{ad}$, where k_0 is an established index.

5. GEODETIC EQUATIONS

Let us write an equation for geodesics $\Gamma \subset P$ with respect to the connection $\omega^{A}{}_{B}$ on $\underline{P}$, i.e., $\nabla_u u = 0$, or

$$u^B \nabla_B u^A = 0 \tag{5.1}$$

where u, $(u^A(t))$ is a tangent vector to the geodesic line, and ∇ means covariant derivative with respect to the connection $\omega^A{}_B$. Using equation (4.8), one easily finds

$$\frac{\bar{D}u^\alpha}{dt}+u^\beta(u^c\rho^2)[l_{cd}g^{\alpha\delta}(2H^d{}_{\beta\delta}-L^d{}_{\beta\delta})-l_{dc}g^{\delta\alpha}L^d{}_{\delta\beta}]+u^bu^cN^\alpha{}_{cb}=0 \quad (5.2)$$

$$\frac{du^a}{dt}+u^\beta u^\gamma L^a{}_{\gamma\beta}-\frac{1}{\rho^2}u^\beta u^c(g_{\delta\beta}l^{ba}N^\delta{}_{bc}+g_{\beta\delta}l^{ab}N^\delta{}_{cb})+u^bu^c\tilde{\Gamma}^a{}_{cb}=0 \quad (5.3)$$

One easily transforms (5.3) into

$$\frac{1}{\rho^2}\frac{d(\rho^2u^a)}{dt}-\frac{2}{\rho}\rho_{,\beta}u^\beta u^a$$
$$-\frac{1}{\rho^2}u^\beta u^c(g_{\delta\beta}l^{ba}N^\delta{}_{bc}+g_{\beta\delta}l^{ab}N^\delta{}_{cb})$$
$$+u^bu^c\tilde{\Gamma}^a{}_{cb}+u^\beta u^\gamma L^a{}_{\gamma\beta}=0 \quad (5.4)$$

or

$$\left(\frac{1}{\rho^2}\frac{d(\rho^2u^a)}{dt}+u^bu^c\tilde{\Gamma}^a{}_{cb}+u^\beta u^\gamma L^a{}_{\gamma\beta}\right)$$
$$-\frac{1}{\rho^2}[(\rho^2)_{,\beta}u^\beta\delta^a{}_cu^c+u^\beta u^c(g_{\delta\beta}l^{ba}N^\delta{}_{bc}+g_{\beta\delta}l^{ab}N^\delta{}_{cb})]=0 \quad (5.5)$$

where

$$\frac{d\rho}{dt}=\rho_{,\beta}u^\beta \quad (5.6)$$

$\bar{D}/dt$ means covariant derivative with respect to $\bar{\omega}^\alpha{}_\beta$ along the line to which $u^\alpha(t)$ is tangent. In the symmetric Kaluza-Klein theory or in a five-dimensional Jordan-Thiry theory, $2\rho^2u^b$ (see Section 3 and ref. 22) has the interpretation of (q^b/m_0) for a test particle (q^b is the color or isotopic charge of the test particle) and the system of equations (5.1)-(5.3) has the first integrals $u^b\rho^2=\text{const}$. In our case it is possible iff

$$L^a{}_{\rho\beta}=-L^a{}_{\beta\rho} \quad (5.7)$$

$$\tilde{\Gamma}^a{}_{cb}=\tilde{\Gamma}^a{}_{bc} \quad (5.8)$$

and

$$(\rho^2)_{,\beta}u^\beta\delta^a{}_cu^c+u^\beta u^c(g_{\delta\beta}l^{ba}N^\delta{}_{bc}+g_{\beta\delta}l^{ab}N^\delta{}_{cb})=0 \quad (5.9)$$

Using (5.9) and (4.9), we easily get

$$N^\delta{}_{ac}=-\rho\tilde{g}^{(\delta\beta)}\rho_{,\beta}l_{ac}=\Gamma^\delta{}_{ac} \quad (5.10)$$

where

$$\tilde{g}^{(\alpha\beta)}g_{(\alpha\gamma)} = \delta^{\beta}{}_{\gamma}$$

$\tilde{g}^{(\alpha\beta)}$ is an inverse tensor for $g_{(\alpha\beta)}$ and

$$\Gamma^{a}{}_{\alpha c} = \frac{1}{\rho} g_{\alpha\delta}\tilde{g}^{(\delta\beta)}\rho_{,\beta}\delta^{a}{}_{c} \tag{5.11}$$

$$\Gamma^{b}{}_{c\beta} = \frac{1}{\rho} g_{\delta\beta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\delta^{b}{}_{c} \tag{5.12}$$

Using (5.10), (5.7), and (5.8), we transform (5.4) and (5.5) into

$$\frac{\bar{D}u^{\alpha}}{dt} + \left(\frac{q^{c}}{m_0}\right)u^{\beta}\left[l_{cd}g^{\alpha\delta}H^{d}{}_{\beta\phi} - \frac{1}{2}(l_{cd}g^{\alpha\delta} - l_{dc}g^{\delta\alpha})L^{d}{}_{\beta\delta}\right] - \frac{1}{4\rho^3}\tilde{g}^{(\alpha\beta)}\rho_{,\beta}\left(\frac{h_{bc}q^{b}q^{c}}{m_0}\right) = 0 \tag{5.13}$$

$$\frac{d}{dt}\left(\frac{q^{a}}{m_0}\right) = 0 \tag{5.14}$$

or

$$\frac{\bar{D}u^{\alpha}}{dt} + \left(\frac{q^{c}}{m_0}\right)u^{\beta}\left(l_{cd}g^{\alpha\delta}H^{d}{}_{\beta\delta} - \frac{1}{2}(l_{cd}g^{\alpha\delta} - l_{dc}g^{\delta\alpha})L^{d}{}_{\beta s}\right) - \frac{\|q\|^2}{8m_0^2}\tilde{g}^{(\alpha\beta)}\left(\frac{1}{\rho^2}\right)_{,\beta} = 0 \tag{5.15}$$

$$\frac{d}{dt}\left(\frac{q^{b}}{m_0}\right) = 0 \tag{5.16}$$

[$q/m_0 = 2\hat{\varkappa}(\mathrm{ver}(u(t)))$; see Appendix B], where $\|q\| = (-h_{ab}q^{a}q^{b})^{1/2}$ is a length of color (isotopic) charge in the Lie algebra $\mathfrak{g}$ of the group G [in color (isotopic) space]. $\|q\|^2$ is positive defined if G is compact. Usually h_{ab} (the Killing-Cartan tensor) is negative defined in the case of semisimple compact Lie algebras. It seems that $\|q\|^2$ could be connected to the Casimir operator of some representations of the group G. If $\rho = \text{const}$ $(=1)$, we get the equation

$$\frac{\bar{D}u^{\alpha}}{dt} - 2u^{b}\left(l_{bd}g^{\alpha\beta}H^{d}{}_{\beta\gamma} - \frac{1}{2}(l_{bd}g^{\alpha\beta} - l_{db}g^{\beta\alpha})L^{d}{}_{\beta\gamma}\right)u^{\gamma} = 0$$

$$\frac{du^{b}}{dt} = 0 \tag{5.17}$$

This equation, in the case of the symmetric metric $g_{\alpha\beta}=g_{\beta\alpha}$, $l_{ab}=h_{ab}=h_{ba}$, turns into

$$\frac{\bar{D}u^{\alpha}}{dt}-2u^{b}h_{bd}g^{\alpha\beta}H^{d}{}_{\beta\gamma}u^{\gamma}=0$$
$$\frac{du^{b}}{dt}=0,\qquad \frac{1}{m_0}=2\,\mathrm{ver}(u(t)) \tag{5.18}$$

Sometimes it is convenient to consider q as an element of the Lie algebra of G, $(\mathfrak{g})$. In this case we define

$$q=q^{a}X_{a}=2m_0[\hat{\varkappa}(\mathrm{ver}(u(t)))]^{a}X_{a}$$

Equation (5.17) is called the Wong equation (see refs. 7 and 92) in the case of $G=SU(2)$ and contains the Lorentz force term for the Yang–Mills field. From the historical point of view this equation should be called rather the Kerner equation, because it appeared for the first time in Kerner's paper (see ref. 7) in curved space-time for an arbitrary semisimple gauge group. Thus, we get in the first equation of (5.17) a Lorentz-like force term in the case of the nonsymmetric metric for an arbitrary gauge field. Our equation (5.13) or (5.15) contains two more terms. The first term

$$-\frac{1}{2}\left(\frac{q^{c}}{m_0}\right)u^{\beta}(l_{cd}g^{\alpha\delta}-l_{dc}g^{\delta\alpha})L^{d}{}_{\beta\delta} \tag{5.19}$$

is known from the nonsymmetric non-Abelian Kaluza–Klein theory and it vanishes if both metrics on space-time $g_{\alpha\beta}$ and on a typical fiber l_{ab} become symmetric. The second term,

$$-\frac{1}{8}\frac{\|q\|^{2}}{m_0^{2}}\tilde{g}^{(\alpha\beta)}\left(\frac{1}{\rho^{2}}\right)_{,\beta}$$

describes interaction with the scalar force for the test particle.

Thus, we formulate the following theorem.

Theorem III.

1. Let conditions 1–4 from the Theorem I be satisfied and let $P=\rho^{2}l$.
2. Let there be an Ad-type field of 2-forms on $\underline{P}$, $L=\frac{1}{2}L^{a}{}_{\mu\nu}\theta^{\mu}\wedge\theta^{\nu}X_{a}$, with values in the Lie algebra of G, $(\mathfrak{g})$, such that (4.10) is satisfied.
3. Let there be on $\underline{P}$ an Ad-type field of 2-forms, $\Gamma=\frac{1}{2}\Gamma^{a}{}_{bc}\theta^{b}\wedge\theta^{c}X_{a}$, with values in the Lie algebra of G such that (4.11) is satisfied. Then there is one and only one connection $\tilde{\omega}$ on the bundle of frames over $\underline{P}$ such that the geodetic equation with respect to it possesses n first integrals of motion that are Ad-type quantities on $\underline{P}$ (a gauge bundle).

In order to understand better the physical content of geodetic equations, we should project it on a space-time E. In the electromagnetic case it is very easy because $F_{\mu\nu}$, $H_{\mu\nu}$, and ρ are well defined on a space-time E. This is of course true in any Abelian case. In a general non-Abelian case we should reformulate them in a gauge-dependent manner. Let us consider a section $f: E \to \underline{P}$ [i.e., $f \to r(f)$]. We can define a gauge-dependent projection, $t \to g(t) \in G$, of a curve, $t \to X(t)$ [given by equations (5.15)-(5.16)] such that

$$X(t) = r(f(t))g(t) \tag{5.20}$$

and define gauge-dependent quantities

$$F^a{}_{\mu\nu} = f^* H^a{}_{\mu\nu} \tag{5.21a}$$

$$B^a{}_{\mu\nu} = f^* L^a{}_{\mu\nu} \tag{5.21b}$$

We easily get

$$H^a{}_{\mu\nu}(X(t)) = U^a{}_{a'}(g(t)^{-1}) F^{a'}{}_{\mu\nu}(f(t)) \tag{5.22a}$$

$$L^a{}_{\mu\nu}(X(t)) = U^a{}_{a'}(g(t)^{-1}) B^{a'}{}_{\mu\nu}(f(t)) \tag{5.22b}$$

where U is an adjoint representation. Let us define similarly as in ref. 9 a gauge-dependent charge Q,

$$Q^a(t) = U^a{}_{a'}(g(t)^{-1}) q^{a'} \tag{5.23}$$

$[Q(t) = 2m_0 \operatorname{Ad}_G(g(t)^{-1})\varkappa(\operatorname{ver}(u(t)))$; see Appendix B]. Then let us define a gauge-dependent tensor m_{ab},

$$m_{ab}(t) = U^{a'}{}_a(g(t))\, U^{b'}{}_b(g(t))\, l_{a'b'} \tag{5.24}$$

Then the geodetic equations (5.13) and (5.14) can be rewritten in an equivalent form

$$\frac{\bar{D}u^\alpha}{dt} + \left(\frac{Q^c}{m_0}\right) u^\beta \left[m_{cd} g^{\alpha\delta} F^d{}_{\beta\delta} - \frac{1}{2}(m_{cd} g^{\alpha\delta} - m_{dc} g^{\delta\alpha}) B^d{}_{\beta\delta} \right]$$
$$- \frac{\|Q\|^2}{8m_0^2} \tilde{g}^{(\alpha\beta)} \left(\frac{1}{\rho^2}\right)_{,\beta} = 0 \tag{5.25}$$

$$\frac{dQ^a}{dt} - C^a{}_{cb} Q^c A^b{}_\nu u^\nu = 0 \tag{5.26}$$

where $f^*\omega = A^b{}_\mu \bar{\theta}^\mu X_b$ is a four-potential.

Sometimes it is convenient to consider Q as an element of the Lie algebra of G. In this case we define

$$Q(t) = Q^a(t) X_a = e^* q = 2m_0[\operatorname{Ad}_G(g^{-1}(t))\hat{\varkappa}(\operatorname{ver}(u(t)))]^a X_a$$

Thus, we get the gauge-dependent form of a generalized Kerner–Wong equation. The second equation, i.e., (5.26), is exactly the same as in the symmetric Kaluza-Klein theory and should be called the Kopczynski equation, for it appears for the first time in ref. 9. The gauge-dependent charge Q is covariantly constant and in general not constant in non-Abelian theories. The gauge-independent charge q is constant in Abelian and non-Abelian theories. It is a first integral of motion. For this we will examine the general properties of geodetic equations as equations of motion for a test particle using equations (5.15)–(5.16). Let us calculate the length of a gauge-dependent charge

$$\begin{aligned}\|Q\|^2 &= -h_{ab}Q^aQ^b \\ &= -h_{a'b'}U^{a'}{}_a(g(t)^{-1})U^{b'}{}_b(g(t)^{-1})q^aq^b \\ &= -h_{ab}q^aq^b = \|q\|^2\end{aligned} \tag{5.27}$$

Thus, the gauge-dependent charge has a constant length. This result can be obtained directly from equation (5.26).

Let us consider the equation $\nabla_{u(t)}u(t)=0$ in a more geometrical way i.e., for horizontal and vertical parts of $u(t)$ [horizontality is understood in a sense of the connection ω on $\underline{P}$ (a gauge bundle)]. One has

$$\begin{aligned}\operatorname{hor}(\nabla_{u(t)}u(t))=0 \\ \operatorname{ver}(\nabla_{u(t)}u(t))=0\end{aligned} \tag{5.3*}$$

One gets

$$\begin{aligned}\operatorname{hor}(\nabla_{u(t)}\operatorname{hor}(u(t))+\nabla_{u(t)}\operatorname{ver}(u(t)))=0 \\ \operatorname{ver}(\nabla_{(t)}\operatorname{ver}(u(t))+\nabla_{u(t)}\operatorname{hor}(u(t)))=0\end{aligned} \tag{5.3**}$$

and

$$\begin{aligned}&\operatorname{hor}(\nabla_{\operatorname{hor}(u(t))}\operatorname{hor}(u(t))+\nabla_{\operatorname{ver}(u(t))}\operatorname{hor}(u(t))+\nabla_{\operatorname{ver}(u(t))}\operatorname{ver}(u(t)) \\ &\quad+\nabla_{\operatorname{hor}(u(t))}\operatorname{ver}(u(t)))=0 \\ &\operatorname{ver}(\nabla_{\operatorname{hor}(u(t))}\operatorname{ver}(u(t))+\nabla_{\operatorname{ver}(u(t))}\operatorname{ver}(u(t))+\nabla_{\operatorname{hor}(u(t))}\operatorname{ver}(u(t)) \\ &\quad+\nabla_{\operatorname{ver}(u(t))}\operatorname{hor}(u(t)))=0\end{aligned} \tag{5.3***}$$

Taking $v(t)=\hat{\varkappa}(\operatorname{ver}(u(t)))$, one gets

$$\begin{aligned}&\operatorname{hor}(\nabla_{\operatorname{hor}(u(t))}\operatorname{hor}(u(t))+\nabla_{\hat{\varkappa}^{-1}(v(t))}\operatorname{hor}(u(t))+\nabla_{\hat{\varkappa}^{-1}(v(t))}\hat{\varkappa}^{-1}(v(t)) \\ &\quad+\nabla_{\operatorname{hor}(u(t))}\hat{\varkappa}^{-1}(v(t)))=0 \\ &\operatorname{ver}(\nabla_{\operatorname{hor}(u(t))}\hat{\varkappa}^{-1}(v(t))+\nabla_{\hat{\varkappa}^{-1}(v(t))}\hat{\varkappa}^{-1}(v(t))+\nabla_{\operatorname{hor}(u(t))}\hat{\varkappa}^{-1}(v(t)) \\ &\quad\nabla_{\hat{\varkappa}^{-1}(v(t))}\operatorname{hor}(u(t)))=0\end{aligned} \tag{5.3'}$$

Supposing

$$\frac{dv}{dt}(t)=0 \qquad \text{and} \qquad R^*(g)\hat{\varkappa}=L^*(g)\hat{\varkappa}=\hat{\varkappa}$$

we get conditions imposed on the connection $\omega^A{}_B$, $(\tilde{\omega})$ (see Appendix A).

Finally, we define normal coordinates on P. Let $\exp: T(\underline{P}) \to \underline{P}$ be the exponential map on (P, γ), such that $\exp_p: \mathrm{Tan}_p(P) \to P$ for each $p \in P$, $\exp_p(V) = \Gamma_v(1)$, where $\Gamma_v(1)$ is an endpoint of a segment of a geodesic through p whose tangent at p is V for an arc parameter equal to 1. Choosing an orthonormal basis $\{e_A\}$ for $\mathrm{Tan}_p(P)$, we define a coordinate system in the neighborhood of P assigning to the point $\exp_p(\sum_A x^A e_A)$ the coordinates $(x^1, x^2, \ldots, x^{n+4})$. We call them normal coordinates. It is easy to see that the physical interpretation of normal coordinates is the following. They are initial velocities and gauge-independent charges of test particles in such a way that

$$x^a = \frac{1}{2\rho^2}\left(\frac{q^a}{m_0}\right) \qquad \text{and} \qquad x^\alpha = u_0^\alpha$$

We can also define the function s,

$$\pm s^2 = (x^1)^2 - \sum_{A=2}^{n+4} (x^A)^2$$

and polar coordinates s, $\theta_1, \theta_2, \ldots, \theta_{n+3}$. In the case of spacelike geodesics our interpretation breaks down (as trajectories of ordinary test particles). They are in this case tachyons. Moreover, supposing that $u_0{}^\alpha$ is an initial velocity of a tachyon, we can maintain our interpretation.

6. GEOMETRY ON THE MANIFOLD P

Using (4.8) and (5.10), (5.11), and (5.12), one easily writes the connection $\omega^A{}_B$ on $\underline{P}$:

$$\omega^A{}_B = \left(\begin{array}{c|c} \pi^*(\bar{\omega}^\alpha{}_\beta) - \rho^2 l_{db} g^{\delta\alpha} L^d{}_{\delta\beta}\theta^b & L^a{}_{\beta\gamma}\theta^\gamma + (1/\rho) g_{\beta\delta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\theta^a \\ \hline \rho^2 l_{bd} g^{\alpha\beta}(2H^d{}_{\gamma\beta} - L^d{}_{\gamma\beta})\theta^\gamma - \rho\tilde{g}^{(\alpha\beta)}\rho_{,\beta} l_{bc}\theta^c & (1/\rho) g_{\delta\beta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\delta^a{}_b\theta^\beta + \tilde{\omega}^a{}_b \end{array}\right) \tag{6.1}$$

where $L^a{}_{\alpha\beta} = -L^a{}_{\beta\alpha}$ is a tensor of Ad type on $\underline{P}$ such that

$$l_{dc} g_{\mu\beta} g^{\gamma\mu} L^d{}_{\gamma\alpha} + l_{cd} g_{\alpha\mu} g^{\mu\gamma} L^d{}_{\beta\gamma} = 2 l_{cd} g_{\alpha\dot{\mu}} g^{\mu\gamma} H^d{}_{\beta\gamma} \tag{6.2}$$

$\tilde{\omega}^a{}_b = \tilde{\Gamma}^a{}_{bc}\theta^c$ is a connection on an internal space (typical fiber) compatible with the metric l_{ab} such that

$$l_{db}\tilde{\Gamma}^d{}_{ac} + l_{ad}\tilde{\Gamma}^d{}_{cb} = -l_{db} C^d{}_{ac} \tag{6.3}$$

$$\tilde{\Gamma}^a{}_{bc} = -\tilde{\Gamma}^a{}_{cb} \tag{6.4}$$

The connection $\tilde{\omega}^a{}_b$ on the typical fiber is an analogue of the connection $\bar{\omega}^\alpha{}_\beta$ on space-time. Thus, we suppose for the sake of symmetry that

$$\tilde{Q}^a{}_{ba}(\tilde{\Gamma}) = 0 \tag{6.5}$$

This means that

$$\tilde{\Gamma}^a{}_{ba} = 0 \tag{6.6}$$

$\tilde{g}^{(\alpha\beta)}$ is an inverse tensor for the symmetric part of the metric $g_{(\alpha\beta)}$,

$$\tilde{g}^{(\alpha\beta)} g_{(\alpha\gamma)} = \delta^\alpha{}_\gamma \tag{6.7}$$

Now we introduce the second connection

$$W^A{}_B = \omega^A{}_B - \frac{4}{3(n+2)} \delta^A{}_B \bar{W} \tag{6.8}$$

where

$$\bar{W} = \bar{W}_\nu \theta^\nu = \tfrac{1}{2}(\bar{W}^\sigma{}_{\nu\sigma} - \bar{W}^\sigma{}_{\sigma\nu})\theta^\nu$$

and $n = \dim G$. It is easy to see that $\bar{W}$ is a horizontal 1-form

$$\text{hor } \bar{W} = \bar{W} \tag{6.9}$$

Horizontality is understood here in the sense of the connection ω (connection on the fiber bundle $\underline{P}$ over E with the structural group G). The connection is right-invariant with respect to the right action of the group G on $\underline{P}$.

Thus, we have now all $(n+4)$-dimensional analogues from Moffat's theory of gravitation: two connections $\omega^A{}_B$ and $W^A{}_B$ and the nonsymmetric metric γ_{AB}. Now let us turn to calculations of the torsion for $\omega^A{}_B$,

$$\Theta^A(\Gamma) = D\theta^A \tag{6.10}$$

where D is the exterior covariant derivative with respect to the connection $\omega^A{}_B$. One easily finds

$$Q^\alpha{}_{\beta\gamma}(\Gamma) = \bar{Q}^\alpha{}_{\beta\gamma}(\bar{\Gamma}) \tag{6.11}$$

$$Q^\alpha{}_{\gamma\beta}(\Gamma) = -Q^\alpha{}_{b\gamma}(\Gamma)$$
$$= 2\rho^2 l_{bd} g^{\alpha\beta} H^d{}_{\gamma\beta} + \rho^2(l_{bd} g^{\alpha\beta} + l_{db} g^{\beta\alpha}) L^d{}_{\beta\gamma} \tag{6.12}$$

$$Q^a{}_{\mu\nu}(\Gamma) = 2(H^a{}_{\mu\nu} - L^a{}_{\mu\nu}) = -2K^a{}_{\mu\nu} \tag{6.13}$$

$$Q^a{}_{\beta b}(\Gamma) = -Q^a{}_{b\beta}(\Gamma) = \frac{1}{\rho} g_{\delta\beta} \tilde{g}^{(\delta\gamma)} \rho_{,\gamma} \delta^a{}_b \tag{6.14}$$

$$Q^\alpha{}_{bc}(\Gamma) = 2\rho \tilde{g}^{(\alpha\beta)} \rho_{,\beta} l_{[bc]} = 2\mu\rho \tilde{g}^{(\alpha\beta)} \rho_{,\beta} k_{bc} \tag{6.15}$$

$$Q^a{}_{bc}(\Gamma) = \tilde{Q}^a{}_{bc}(\tilde{\Gamma}) = -(C^a{}_{bc} + 2\tilde{\Gamma}^a{}_{bc}) \tag{6.16}$$

where $\bar{Q}^{\alpha}{}_{\beta\gamma}(\bar{\Gamma})$ is the torsion of the connection $\bar{\omega}^{\alpha}{}_{\beta}$ and $\bar{Q}^{a}{}_{bc}(\tilde{\Gamma})$ is the torsion of the connection $\tilde{\omega}^{a}{}_{b}$. We will find later the physical interpretation of the tensor $K^{a}{}_{\mu\nu}$ on $\underline{P}$, which is of course of Ad type. Equation (6.15) gives us an independent interpretation of the tensor k_{ab} as a factor in a torsion tensor $Q^{\alpha}{}_{ab}$. Let us turn to a calculation of the 2-form of curvature for the connection $\omega^{A}{}_{B}$. We have

$$\Omega^{A}{}_{B}(\Gamma)=d\omega^{A}{}_{B}+\omega^{A}{}_{C}\wedge\omega^{C}{}_{B} \tag{6.17}$$

One easily gets

$$\begin{aligned}\Omega^{\alpha}{}_{\beta}(\Gamma)=&\,\bar{\Omega}^{\alpha}{}_{\beta}(\bar{\Gamma})+\rho^{2}[l_{cd}g^{\alpha\omega}(2H^{d}{}_{\omega[\mu}-L^{d}{}_{\omega[\mu})L^{c}{}_{\nu]\beta}\\&-l_{db}g^{\delta\alpha}L^{d}{}_{\delta\beta}H^{b}{}_{\mu\nu}]\theta^{\mu}\wedge\theta^{\nu}+[\rho l_{bd}g^{\alpha\omega}g_{\beta\delta}\tilde{g}^{(\delta\xi)}\rho_{,\xi}(2H^{d}{}_{\mu\omega}-L^{d}{}_{\mu\omega})\\&-\bar{\nabla}_{\mu}(\rho^{2}l_{db}g^{\delta\alpha}L^{d}{}_{\delta\beta})+\rho\tilde{g}^{(\alpha\omega)}\rho_{,\omega}l_{cb}L^{c}{}_{\beta\mu}]\theta^{\mu}\wedge\theta^{b}\\&+\left(\rho^{4}l_{d[b}l_{|e|f]}g^{\delta\alpha}g^{\xi\gamma}L^{d}{}_{\delta\gamma}L^{e}{}_{\xi\beta}+\frac{\rho^{2}}{2}l_{dp}g^{\delta\alpha}L^{d}{}_{\delta\beta}C^{p}{}_{bf}\right.\\&\left.+\tilde{g}^{(\alpha\omega)}\rho_{,\omega}g_{\beta\delta}\tilde{g}^{(\delta\xi)}\rho_{,\xi}l_{[bf]}\right)\theta^{b}\wedge\phi^{f}\end{aligned} \tag{6.18}$$

$$\begin{aligned}\Omega^{\alpha}{}_{b}(\Gamma)=&\left\{\bar{\nabla}_{[\mu}[\rho^{2}l_{bd}g^{\alpha\beta}(2H^{d}{}_{\nu]\beta}-L^{d}{}_{\nu]\beta})]\right.\\&+\frac{\rho^{2}}{2}l_{bd}g^{\alpha\beta}(2H^{d}{}_{\gamma\beta}-L^{d}{}_{\gamma\beta})\bar{Q}^{\gamma}{}_{\mu\nu}(\bar{\Gamma})-\rho\tilde{g}^{(\alpha\beta)}\rho_{,\beta}l_{bc}H^{c}{}_{\mu\nu}\\&\left.+\rho l_{bd}g^{\alpha\beta}(2H^{d}{}_{[\mu|\beta|}-L^{d}{}_{[\mu|\beta|})g_{|\delta|\nu]}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\right\}\theta^{\mu}\wedge\theta^{\nu}\\&+\{\bar{\nabla}_{\mu}(\rho\tilde{g}^{(\alpha\beta)}\rho_{,\beta})l_{ba}+\tilde{\bar{\nabla}}_{a}[\rho^{2}l_{bd}(2H^{d}{}_{\mu\beta}-L^{d}{}_{\mu\beta})g^{\alpha\delta}]\\&-\rho^{4}l_{da}b_{bf}g^{\delta\alpha}g^{\gamma\beta}L^{d}{}_{\delta\gamma}(2H^{f}{}_{\mu\beta}-L^{f}{}_{\mu\beta})\\&-\tilde{g}^{(\alpha\xi)}\rho_{,\xi}g_{\delta\mu}\tilde{g}^{(\delta\nu)}\rho_{,\nu}l_{ba}\}\theta^{a}\wedge\theta^{\mu}\\&+\left[\frac{\rho}{2}\tilde{g}^{(\alpha\beta)}\rho_{,\beta}l_{bd}C^{d}{}_{ca}+\rho^{3}\tilde{g}^{(\gamma\beta)}\rho_{,\beta}g^{\delta\alpha}L^{d}{}_{\delta\gamma}l_{d[a}l_{|b|c]}\right]\theta^{c}\wedge\theta^{a}\end{aligned} \tag{6.19}$$

$$\begin{aligned}\Omega^{a}{}_{\beta}(\Gamma)=&\left(\bar{\nabla}_{[\mu}L^{a}{}_{|\beta|\nu]}+\frac{1}{2}L^{a}{}_{\beta\gamma}\bar{Q}^{\gamma}{}_{\mu\nu}(\bar{\Gamma})+\frac{1}{\rho}g_{\beta\delta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}H^{a}{}_{\mu\nu}\right.\\&\left.+\frac{1}{\rho}g_{\delta[\mu}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}L^{a}{}_{|\beta|\gamma]}\right)\theta^{\mu}\wedge\theta^{\nu}\\&+\left[\bar{\nabla}_{\gamma}\left(\frac{1}{\rho}g_{\beta\delta}\tilde{g}^{(\delta\mu)}\rho_{,\mu}\right)\delta^{a}{}_{c}-\tilde{\bar{\nabla}}_{c}L^{a}{}_{\beta\gamma}-\rho^{2}l_{db}g^{\delta\mu}L^{a}{}_{\mu\gamma}L^{d}{}_{\delta\beta}\right.\end{aligned}$$

$$+\frac{1}{\rho^2} g_{\delta\gamma}\tilde{g}^{(\delta\beta)}\rho_{,\beta}g_{\beta\nu}\tilde{g}^{(\nu\mu)}\rho_{,\mu}\delta^a{}_b\Big]\theta^\gamma\wedge\theta^b$$
$$+\left(\tilde{\nabla}_{[b}\left(\frac{1}{\rho}g_{\beta\delta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\right)\delta^a{}_{c]}-\frac{1}{2\rho}g_{\beta\delta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}C^a{}_{bc}\right.$$
$$\left.+\rho g_{\gamma\nu}\tilde{g}^{(\nu\mu)}\rho_{,\mu}g^{\delta\gamma}L^d{}_{\delta\beta}l_{d[b}\delta^a{}_{c]}\right)\theta^b\wedge\theta^c \tag{6.20}$$

$$\Omega^a{}_b(\Gamma)=\tilde{\Omega}^a{}_b(\tilde{\Gamma})+\Bigg[-\delta^a{}_b\bar{\nabla}_{[\nu}\left(\frac{1}{\rho}g_{|\delta|\mu]}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\right)$$
$$+\frac{1}{2\rho}\delta^a{}_b g_{\delta\alpha}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\bar{Q}^\alpha{}_{\mu\nu}(\bar{\Gamma})$$
$$+\rho^2 l_{bd}g^{\gamma\beta}L^d{}_{\gamma[\mu}(2H^d{}_{\nu]\beta}-L^d{}_{\nu]\beta})\Bigg]\theta^\mu\wedge\theta^\nu$$
$$+[-\rho\tilde{g}^{(\gamma\beta)}\rho_{,\beta}l_{bc}L^a{}_{\gamma\mu}-\rho l_{bd}\tilde{g}^{(\beta\nu)}\rho_{,\nu}(2H^d{}_{\mu\beta}-L^d{}_{\mu\beta})\delta^a{}_c]\theta^\mu\wedge\theta^c$$
$$-g_{\gamma\delta}\tilde{g}^{(\delta\nu)}\rho_{,\nu}l_{b[c}\delta^a{}_{d]}\theta^d\wedge\theta^c \tag{6.21}$$

where $\bar{\nabla}_\mu$ means the covariant derivative with respect to the connection $\bar{\omega}^\alpha{}_\beta$. Here $\bar{\Omega}^\alpha{}_\beta(\bar{\Gamma})$ is the 2-form of curvature for the connection $\bar{\omega}^\alpha{}_\beta$ on the space-time E and $\tilde{\Omega}^a{}_b(\tilde{\Gamma})$ is the 2-form of curvature for the connection $\tilde{\omega}^a{}_b$ on the typical fiber; $\tilde{\nabla}$ means its covariant derivative. $\bar{Q}^\alpha{}_{\beta\gamma}(\bar{\Gamma})$ is the torsion of the connection $\bar{\omega}^\alpha{}_\beta$ on E. Using (6.18)–(6.21), one easily reads the tensor of curvature for the connection $\omega^A{}_B$,

$$R^\alpha{}_{\beta\mu\nu}(\Gamma)=\bar{R}^\alpha{}_{\beta\mu\nu}(\bar{\Gamma})$$
$$+2\rho^2[l_{cd}g^{\alpha\omega}(2H^d{}_{w[\mu}-L^d{}_{\omega[\mu})L^c{}_{\nu]\beta}+l_{db}g^{\delta\alpha}L^d{}_{\delta\beta}H^b{}_{\mu\nu}] \tag{6.22a}$$
$$R^\alpha{}_{\beta\mu b}(\Gamma)=-R^\alpha{}_{\beta b\mu}(\Gamma)$$
$$=\rho l_{bd}g^{\alpha\omega}g_{\beta\delta}\tilde{g}^{(\delta\xi)}\rho_{,\xi}(2H^d{}_{\mu\omega}-L^d{}_{\mu\omega})$$
$$-\bar{\nabla}_\mu(\rho^2 l_{db}g^{\delta\alpha}L^d{}_{\delta\beta})-\rho\tilde{g}^{(\alpha\omega)}\rho_{,\omega}l_{cd}L^c{}_{\beta\mu} \tag{6.22b}$$
$$R^\alpha{}_{\beta bf}(\Gamma)=2\rho^4 l_{d[b}l_{|e|f]}g^{\delta\alpha}g^{\xi\gamma}L^d{}_{\delta\gamma}L^e{}_{\xi\beta}$$
$$+\tilde{g}^{(\alpha\omega)}\rho_{,\omega}g_{\beta\delta}\tilde{g}^{(\delta\xi)}\rho_{,\xi}l_{[bf]}+\frac{\rho^2}{2}l_{dp}g^{\delta\alpha}L^d{}_{\delta\beta}C^p{}_{bf} \tag{6.22c}$$

$$R^\alpha{}_{b\mu\nu}(\Gamma)=2\bar{\nabla}_{[\mu}[\rho^2 l_{bd}g^{\alpha\beta}(2H^d{}_{\nu]\beta}-L^d{}_{\nu]\beta})]$$
$$+\rho^2 l_{bd}g^{\alpha\beta}(2H^d{}_{\gamma\beta}-L^d{}_{\gamma\beta})\bar{Q}^\gamma{}_{\mu\nu}(\bar{\Gamma})-\rho\tilde{g}^{(\alpha\beta)}\rho_{,\beta}l_{bc}H^c{}_{\mu\nu}$$
$$+2\rho l_{bd}g^{\alpha\beta}(2H^d{}_{[\mu|\beta|}-L^d{}_{[\mu|\beta|})g_{|\delta|\nu]}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma} \tag{6.22d}$$
$$R^\alpha{}_{ba\mu}(\Gamma)=-R^\alpha{}_{b\mu a}(\Gamma)$$
$$=\bar{\nabla}_\mu(\rho\tilde{g}^{(\alpha\beta)}\rho_{,\beta})l_{ba}+\tilde{\nabla}_a[\rho^2 l_{bd}(2H^d{}_{\mu\beta}-L^d{}_{\mu\beta})]$$
$$-\rho^4 l_{da}l_{bf}g^{\delta\alpha}g^{\gamma\beta}L^d{}_{\delta\gamma}(2H^f{}_{\mu\beta}-L^f{}_{\mu\beta})-\tilde{g}^{(\alpha\xi)}\rho_{,\xi}g_{\delta\mu}\tilde{g}^{(\delta\nu)}\rho_{,\nu}l_{ba} \tag{6.22e}$$

$$R^{\alpha}{}_{bac}(\Gamma)=\rho\tilde{g}^{(\alpha\beta)}\rho_{,\beta}l_{bk}C^{k}{}_{ac}+2\rho^{3}\tilde{g}^{(\gamma\beta)}\rho_{,\beta}g^{\delta\alpha}L^{d}{}_{\delta\gamma}l_{d[a}l_{|b|c]} \tag{6.22f}$$

$$R^{a}{}_{b\mu\nu}(\Gamma)=2\rho^{2}l_{bd}g^{\gamma\sigma}(2H^{d}{}_{[\nu|\sigma|}-L^{d}{}_{[\nu|\sigma|})L^{\alpha}{}_{|\gamma|\mu]}+\frac{1}{\rho}\delta^{a}{}_{b}g_{\delta\alpha}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\bar{Q}^{\alpha}{}_{\mu\nu}(\bar{\Gamma})-2\delta^{a}{}_{b}\bar{\nabla}_{[\nu}\left(\frac{1}{\rho}g_{|\delta|\mu]}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\right) \tag{6.22g}$$

$$R^{a}{}_{b\mu c}(\Gamma)=-R^{a}{}_{bc\mu}(\Gamma)=-\rho\tilde{g}^{(\gamma\beta)}\rho_{,\beta}l_{bc}L^{a}{}_{\gamma\beta}-\rho l_{bd}\tilde{g}^{(\beta\nu)}\rho_{,\nu}(2H^{d}{}_{\mu\beta}-L^{d}{}_{\mu\beta})\delta^{a}{}_{c} \tag{6.22h}$$

$$R^{a}{}_{bdc}(\Gamma)=\tilde{R}^{a}{}_{bdc}(\tilde{\Gamma})-2g_{\gamma\delta}\tilde{g}^{(\delta\nu)}\rho_{,\nu}\tilde{g}^{(\gamma\beta)}\rho_{,\gamma}l_{b[c}\delta^{a}{}_{d]} \tag{6.22i}$$

$$R^{a}{}_{\beta\mu\nu}(\Gamma)=2\bar{\nabla}_{[\mu}L^{a}{}_{|\beta|\nu]}+L^{a}{}_{\beta\gamma}\bar{Q}^{\gamma}{}_{\mu\nu}(\bar{\Gamma})+\frac{2}{\rho}g_{\beta\delta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}H^{a}{}_{\mu\nu}+\frac{2}{\rho}g_{\delta[\mu}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}L^{a}{}_{|\beta|\nu]} \tag{6.22j}$$

$$R^{a}{}_{\beta b\gamma}(\Gamma)=-R^{a}{}_{\beta\gamma b}(\Gamma)=\tilde{\nabla}_{b}L^{a}{}_{\beta\gamma}-\bar{\nabla}_{\gamma}\left(\frac{1}{\rho}g_{\beta\delta}\tilde{g}^{(\delta\mu)}\rho_{,\mu}\right)\delta^{a}{}_{b}+\rho^{2}l_{db}g^{\delta\mu}L^{a}{}_{\mu\gamma}L^{d}{}_{\delta\beta}-\frac{1}{\rho^{2}}g_{\delta\gamma}\tilde{g}^{(\delta\beta)}\rho_{,\beta}g_{\beta\nu}\tilde{g}^{(\nu\mu)}\rho_{,\mu}\delta^{a}{}_{b} \tag{6.22k}$$

$$R^{a}{}_{\beta bc}(\Gamma)=2\tilde{\nabla}_{[b}\left(\frac{1}{\rho}g_{\beta\delta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\right)\delta^{a}{}_{c]}-\frac{1}{\rho}g_{\beta\delta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}C^{a}{}_{bc}+2\rho g_{\gamma\nu}\tilde{g}^{(\nu\mu)}\rho_{,\mu}g^{\delta\gamma}L^{d}{}_{\delta\beta}l_{d[b}\delta^{a}{}_{c]} \tag{6.22l}$$

$\bar{R}^{\alpha}{}_{\beta bc}(\bar{\Gamma})$ is the tensor of curvature for the connection $\bar{\omega}^{\alpha}{}_{\beta}$, and $\tilde{R}^{a}{}_{bcd}(\tilde{\Gamma})$ is the tensor of curvature for the connection $\tilde{\omega}^{a}{}_{b}$. Using (6.7), one easily gets the 2-form of curvature for $W^{A}{}_{B}$,

$$\Omega^{A}{}_{B}(W)=\Omega^{A}{}_{B}(\Gamma)-\frac{4}{3(n+2)}\delta^{A}{}_{B}\,d\bar{W}=\Omega^{A}{}_{B}(\Gamma)-\frac{4}{3(n+2)}\delta^{A}{}_{B}\bar{W}_{[\mu,\nu]}\theta^{\mu}\wedge\theta^{\nu} \tag{6.23}$$

and the tensor of curvature for $W^{A}{}_{B}$,

$$\begin{aligned}R^{\alpha}{}_{\beta\mu\nu}(W)&=R^{\alpha}{}_{\beta\mu\nu}(\Gamma)-\frac{8}{3(n+2)}\delta^{\alpha}{}_{\beta}\bar{W}_{[\mu,\nu]}\\ R^{a}{}_{b\mu\nu}(W)&=R^{a}{}_{b\mu\nu}(\Gamma)-\frac{8}{3(n+2)}\delta^{a}{}_{b}\bar{W}_{[\mu,\nu]}\\ R^{\alpha}{}_{\beta cd}(W)&=R^{\alpha}{}_{\beta cd}(\Gamma)\\ R^{a}{}_{bcd}(W)&=R^{a}{}_{bcd}(\Gamma)\\ R^{a}{}_{\beta\mu b}(W)&=R^{a}{}_{\beta\mu b}(\Gamma)\end{aligned} \tag{6.24}$$

where $R^A{}_{BCD}(W)$ is the tensor of curvature for the connection $W^A{}_B$, and $R^A{}_{BCD}(\Gamma)$ is the tensor of curvature for the connection $\omega^A{}_B$.

Now we pass to the calculations of the Moffat-Ricci curvature scalar for the connection $W^A{}_B$ on the manifold $\underline{P}$. We have

$$R(W)=\gamma^{BC}\left[R^A{}_{BCA}(W)+\frac{1}{2}R^A{}_{ABC}(W)\right]$$
$$=g^{\beta\gamma}\left[R^\alpha{}_{\beta\gamma\alpha}(W)+R^a{}_{\beta\gamma a}(W)+\frac{1}{2}R^\alpha{}_{\alpha\beta\gamma}(W)+\frac{1}{2}R^a{}_{a\beta\gamma}\right]$$
$$+\frac{l^{bc}}{\rho^2}\left[R^\alpha{}_{bc\alpha}(W)+R^a{}_{bca}(W)+\frac{1}{2}R^\alpha{}_{\alpha bc}(W)+\frac{1}{2}R^a{}_{abc}(W)\right] \tag{6.25}$$

Using (6.25) and (6.24a)-(6.241), after some calculations, one gets

$$R(W)=\bar{R}(\bar{W})+\frac{\tilde{R}(\tilde{\Gamma})}{\rho^2}-\rho^2(2l_{cd}H^cH^d-l_{cd}g^{\alpha\omega}g^{\beta\mu}L^c{}_{\alpha\beta}H^d{}_{\omega\mu})$$
$$-\frac{P}{\rho^2}g_{\gamma\delta}\tilde{g}^{(\delta\nu)}\rho_{,\nu}\tilde{g}^{(\gamma\beta)}\rho_{,\gamma}+Q(\rho) \tag{6.26}$$

where

$$Q(\rho)=\frac{n}{2\rho}g_{\delta\beta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}g^{\mu\nu}\tilde{Q}^\beta{}_{\mu\nu}(\bar{\Gamma})+\frac{n}{\rho^2}\bar{\nabla}_\alpha(\rho\tilde{g}^{(\alpha\beta)}\rho_{,\beta})$$
$$+ng^{\beta\gamma}\bar{\nabla}_\gamma\left(\frac{1}{\rho}g_{\beta\delta}\tilde{g}^{(\delta\alpha)}\rho_{,\alpha}\right)$$
$$+\frac{n}{2}g^{\mu\nu}\left[\bar{\nabla}_\mu\left(\frac{1}{\rho}g_{\delta\nu}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\right)-\bar{\nabla}_\nu\left(\frac{1}{\rho}g_{\delta\mu}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\right)\right] \tag{6.27}$$

$$P=l^{[dc]}l_{[dc]}-n(n-1) \tag{6.28}$$

$$H^c=g^{[\mu\nu]}H^c{}_{\mu\nu} \tag{6.29}$$

$\bar{R}(\bar{W})$ is the Moffat-Ricci scalar of curvature for the connection $\bar{W}^\alpha{}_\beta$ on E and $\tilde{R}(\tilde{\Gamma})$ is the Moffat-Ricci scalar of curvature for the connection $\tilde{\omega}^a{}_b$ on a typical fiber. Now let us pass to the calculation of a density for the Moffat-Ricci scalar of curvature

$$\gamma^{1/2}R(W)=(-g)^{1/2}|l|^{1/2}\rho^nR(W)$$
$$=(-g|l|)^{1/2}\left[\rho^n\bar{R}(W)+\frac{\tilde{R}(\tilde{\Gamma})}{4\rho^{2-n}}+8\pi\rho^{n+2}\mathcal{L}_{YM}\right.$$
$$\left.+\rho^{n-2}(m\tilde{g}^{(\gamma\mu)}\rho_{,\gamma}\rho_{,\mu}+n^2g^{[\mu\nu]}g_{\delta\mu}\tilde{g}^{(\delta\gamma)}\rho_{,\nu}\rho_{,\gamma})\right]+\partial_\mu K^\mu \tag{6.30}$$

where

$$\mathscr{L}_{YM} = \frac{-1}{8\pi}(2l_{cd}H^cH^d - l_{cd}g^{\alpha\omega}g^{\beta\mu}L^c{}_{\alpha\beta}L^d{}_{\omega\mu}) \tag{6.31}$$

and

$$m = (l^{[dc]}l_{[dc]} - 3n(n-1)] \tag{6.32}$$

$l = \det(l_{ab})$ and

$$K^\mu = \frac{n}{2}\rho^{n-1}(5\tilde{g}^{(\mu\gamma)} - g^{\nu\mu}g_{\delta\nu}\tilde{g}^{(\delta\gamma)})\rho_{,\gamma}$$

From the variational principle point of view, for the density $\gamma^{1/2}R(W)$, the four-divergence $\partial_\mu K^\mu$ does not play any role. It could play a certain role for some topological problems. Thus, we really only have to deal with $B(W)$;

$$B(W) = (-g|l|)^{1/2}\left[\rho^n\bar{R}(\bar{W}) + \frac{\tilde{R}(\tilde{\Gamma})}{\rho^{2-n}} + 8\pi\rho^{n+2}\mathscr{L}_{YM} + \rho^{n-2}(m\tilde{g}^{(\gamma\nu)} + n^2 g^{[\mu\nu]}g_{\delta\mu}\tilde{g}^{(\delta\gamma)})\rho_{,\gamma}\rho_{,\nu}\right] \tag{6.33}$$

Finally, we write down some identities concerning $H^a{}_{\mu\nu}$ and $L^a{}_{\mu\nu}$ coming from equation (6.2),

$$g^{[\mu\nu]}L^a{}_{\mu\nu} = h^{ac}l_{cp}H^p{}_{\mu\nu}g^{[\mu\nu]} \tag{6.34}$$

$$l_{dc}g^{\sigma\nu}g^{\alpha\mu}L^d{}_{\sigma\alpha}H^c{}_{\mu\nu} + l_{cd}g^{\mu\sigma}g^{\nu\beta}L^d{}_{\beta\sigma}H^c{}_{\mu\nu} = 2l_{cd}g^{\mu\sigma}g^{\nu\beta}H^c{}_{\mu\nu}H^d{}_{\beta\sigma} \tag{6.35}$$

$$l_{dc}g^{\alpha\omega}g^{\beta\mu}L^d{}_{\alpha\beta}L^c{}_{\omega\mu} = l_{cd}g^{\alpha\omega}g^{\beta\mu}L^c{}_{\alpha\beta}L^d{}_{\omega\mu} \tag{6.36}$$

Note that all the formulas presented in this section are valid for an arbitrary right-invariant l_{ab}, i.e., such that its symmetric part is not proportional to the Killing-Cartan tensor on G.

7. CONNECTION $\tilde{\omega}^a{}_b$. COSMOLOGICAL CONSTANT

In $R(W)$ and $B(W)$, $\tilde{R}(\tilde{\Gamma})/\rho^{2-n}$ plays the role of a cosmological term. Let us turn to the calculation of the Moffat-Ricci curvature scalar for the connection $\tilde{\omega}^a{}_b$, i.e., $\tilde{R}(\tilde{\Gamma})$. One can find, using (6.3), (6.4), (4.13), and (6.6),

$$\tilde{R}(\tilde{\Gamma}) = 2l^{bk}C^e{}_{ka}\tilde{\Gamma}^a{}_{eb} + l^{bk}C_{ba}\tilde{\Gamma}^a{}_{ek} + l^{bk}\tilde{\Gamma}^a{}_{fk}\tilde{\Gamma}^f{}_{ba} \tag{7.1}$$

One finally gets

$$\tilde{R}(\tilde{\Gamma}) = 2l^{(ae)}h_{ae} + \tfrac{3}{2}l^{be}\tilde{\Gamma}^p{}_{ke}C^k{}_{pb} \tag{7.2}$$

where $\tilde{\Gamma}^a{}_{bc}$ satisfies compatibility conditions

$$l_{db}\tilde{\Gamma}^d{}_{ac}(\mu)+l_{ad}\tilde{\Gamma}^d{}_{cb}(\mu)=-l_{db}C^d{}_{bc} \tag{7.3}$$

and

$$\tilde{\Gamma}^b{}_{ac}(\mu)=-\tilde{\Gamma}^b{}_{ca}(\mu), \qquad \tilde{\Gamma}^b{}_{ab}(\mu)=0 \tag{7.4}$$

It is easy to see that $\tilde{R}(\tilde{\Gamma})$ is a rational function of μ. However, it is a very hard task to find the exact dependence on μ. Therefore, we do not have an exact solution of (7.3) and (7.4). Moreover, we can find an asymptotic dependence for a very large μ. If $\mu\to\infty$, equation (7.3) turns into

$$k_{db}\tilde{\Gamma}^d{}_{ac}+k_{ad}\tilde{\Gamma}^d{}_{cb}=-k_{db}C^d{}_{bc} \tag{7.5}$$

Thus, in the limit of a very large μ, $\tilde{\Gamma}^a{}_{bc}(\mu)$ goes to the constant $\tilde{\Gamma}^a{}_{bc}$ with respect to μ. On the other hand, we have

$$l^{ab}=\frac{\Delta^{ab}}{\Delta} \tag{7.6}$$

where $\Delta=\det(l_{ab})$ and Δ^{ab} is a cofactor matrix formed from l_{ab}. It is easy to see that Δ is a polynomial of nth order with respect to μ and Δ^{ab} a polynomial of $(n-1)$th order with respect to μ. Thus, we finally get for a very large μ

$$\tilde{R}(\tilde{\Gamma})=\frac{2\Delta^{(ae)}}{\Delta}h_{ae}+\frac{3}{2}\frac{\Delta^{be}}{\Delta}\tilde{\Gamma}^p{}_{ke}C^k{}_{pb}\approx\frac{\text{const}}{\mu}$$

or

$$\tilde{R}(\tilde{\Gamma})\simeq\text{const} \qquad \text{for large } \mu \tag{7.7}$$

It may be possible to find an exact solution of (7.3) and (7.4). In this way we get

$$\tilde{R}(\tilde{\Gamma})=\frac{P_m(\mu)}{Q_{m+1}(\mu)} \quad \text{or} \quad \frac{P_m(\mu)}{Q_m(\mu)} \tag{7.8}$$

where P_m, Q_m, and Q_{m+1} are some polynomials with respect to μ of order m and $m+1$. The P_m and Q_{m+1} (Q_m) do not have common divisors. If the polynomial $P_m(\mu)$ has a real root μ_0, we have

$$\tilde{R}(\tilde{\Gamma})=0 \qquad \text{for} \quad \mu=\mu_0$$

If we suppose that $\tilde{\Gamma}^f{}_{cb}$ has a potential $\Xi^f{}_d$ such that

$$\tilde{\Gamma}^f=\tfrac{1}{2}\tilde{\Gamma}^f{}_{cb}v^c\wedge v^b=d\Xi^f=d(\Xi^f{}_dv^d) \tag{7.9}$$

We can find for $\tilde{\Gamma}^f$

$$\tilde{\Gamma}^d{}_{ab}=-2\Xi^d{}_{[a,b]}-\Xi^d{}_cC^c{}_{ab} \tag{7.10}$$

where $,b$ means an action of the left-invariant fields on the group G. For $\tilde{\Gamma}^d{}_{ab}$ a right-invariant quantity we suppose the same for $\Xi^d{}_e$:

$$\Xi^d{}_{a,b} + C^e{}_{ab}\Xi^d{}_e - C^d{}_{eb}\Xi^e{}_a = 0 \tag{7.11}$$

or

$$\hat{\nabla}_b \Xi^d{}_a = 0 \tag{7.11a}$$

Using (7.11), one easily gets

$$\tilde{\Gamma}^d{}_{ab} = C^e{}_{ab}\Xi^d{}_e - C^d{}_{eb}\Xi^e{}_a + C^d{}_{ea}\Xi^e{}_b \tag{7.12}$$

Equation (7.12) defines $\tilde{\Gamma}^d{}_{ab}$ which satisfies conditions (7.4) identically. However, $\Xi^d{}_e$ should be a solution of equation (7.3) if we substitute equation (7.12).

Note the following. $\tilde{\Gamma}^a{}_{bc}$ has at least $g(n) = n[n^2/2 - \frac{3}{2}n - 1)$ independent coordinates. From equation (7.12) one easily gets that it possesses $f(n) = n^2$ independent coordinates. Thus, we should have $g(n) \geq f(n)$. This is possible only if $n > \frac{1}{2}(3 + \sqrt{17})$, where $n = \dim G$. Thus, we get $\dim G > 4$.

Thus, for $G = SO(3)$ we cannot use the formula (7.12).

In the case of a symmetric $l_{ab} = h_{ab}$ one easily finds

$$\Xi^a{}_b = \tfrac{1}{2}\delta^a{}_b \tag{7.13}$$

One can express $\tilde{R}(\tilde{\Gamma})$ in terms of $\Xi^a{}_b$ and gets

$$\tilde{R} = 2l^{(ae)}h_{ae} + \frac{3(2-n)}{2(n-1)} l^{be}\Xi_{be} - \frac{3}{2(n-1)} l^{be}\Xi_{eb} \tag{7.14}$$

where

$$\Xi_{be} = h_{fb}\Xi^f{}_e \tag{7.15}$$

Let us consider the cosmological term for $\rho = 1$, i.e.,

$$\tilde{R}(\tilde{\Gamma}) = \Phi(\mu) \tag{7.16}$$

In general, $\Phi(\mu)$ has the following form:

$$\Phi(\mu) = \frac{P_m(\mu)}{Q_{m+1}(\mu)} \quad \text{or} \quad \frac{P_m(\mu)}{Q_m(\mu)} \tag{7.17}$$

Thus, we can make the cosmological term to vanish if we choose $\mu = \mu_0$ a real root of P_m. In this way we get the physical interpretation of the dimensionless constant μ connecting it to the cosmological constant. If we come back to the ordinary system of units, the cosmological term will be very big numerically because of the numerical constants in front of it. Thus, if $\tilde{R}(\tilde{\Gamma})$ is of the order of one, the cosmological constant is 10^{127} times bigger than the upper limit from observational data. The only mechanism to avoid this unwanted fact is to make it vanish by choosing an appropriate μ. Sometimes we can also make the cosmological constant as small as we want if we choose μ sufficiently big, i.e., $|\mu| \geq 10^{127}$.

Let us consider the connection $\tilde{\Gamma}^a{}_{bc}(g)$ on G, $g \in G$. Moreover, one has

$$\tilde{\Gamma}^a{}_{bc}(g) = \tilde{\Gamma}^{a'}_{b'c'}(\varepsilon) U^a_{a'}(g^{-1}) U^{b'}_b(g) U^{c'}_c(g) \tag{7.18}$$

where U is an adjoint representation of G (see Appendix A). Thus, in order to find $\tilde{\Gamma}^a{}_{bc}(g)$, it is enough to find $\tilde{\Gamma}^a{}_{bc}(\varepsilon)$ for $l_{ab}(\varepsilon)$, i.e., to solve the equation (7.3) for $g = \varepsilon \in G$.

It is also easy to see that

$$\tilde{R}(\tilde{\Gamma})(g)) = \tilde{R}(\tilde{\Gamma}(\varepsilon)) \tag{7.19}$$

Let us consider equation (7.3) for $G = SO(3)$ as unit element of $SO(3)$. Using (2.24), one gets

$$\begin{aligned}\tilde{\Gamma}^3{}_{33}(\varepsilon) &= \tilde{\Gamma}^3{}_{32}(\varepsilon) = \tilde{\Gamma}^3{}_{23}(\varepsilon) = \tilde{\Gamma}^3{}_{31}(\varepsilon) = \tilde{\Gamma}^3{}_{22}(\varepsilon) = \tilde{\Gamma}^3{}_{13}(\varepsilon) \\ &= \tilde{\Gamma}^1{}_{11}(\varepsilon) = \tilde{\Gamma}^3{}_{11}(\varepsilon) = \tilde{\Gamma}^2{}_{33}(\varepsilon) = \tilde{\Gamma}^2{}_{22}(\varepsilon) = \tilde{\Gamma}^2{}_{21}(\varepsilon) = \tilde{\Gamma}^2{}_{12}(\varepsilon) \\ &= \tilde{\Gamma}^2{}_{11}(\varepsilon) = \tilde{\Gamma}^1{}_{22}(\varepsilon) = \tilde{\Gamma}^1{}_{21}(\varepsilon) = \tilde{\Gamma}^1{}_{33}(\varepsilon) = \tilde{\Gamma}^1{}_{12}(\varepsilon) = 0\end{aligned} \tag{7.20a}$$

$$\begin{aligned}\tilde{\Gamma}^3{}_{21}(\varepsilon) &= -\tilde{\Gamma}^3{}_{12}(\varepsilon) = -\frac{1}{2} \\ \tilde{\Gamma}^3{}_{32}(\varepsilon) &= -\tilde{\Gamma}^2{}_{23}(\varepsilon) = \frac{\mu}{\mu^2+4} \\ \tilde{\Gamma}^2{}_{31}(\varepsilon) &= -\tilde{\Gamma}^2{}_{13}(\varepsilon) = \frac{\mu^2+2}{\mu^2+4} \\ \Gamma^1{}_{32}(\varepsilon) &= -\tilde{\Gamma}^1{}_{23}(\varepsilon) = -\frac{2}{\mu^2+4} \\ \tilde{\Gamma}^1{}_{31}(\varepsilon) &= -\tilde{\Gamma}^1{}_{13}(\varepsilon) = -\frac{\mu}{\mu^2+4}\end{aligned} \tag{7.20b}$$

Using equations (7.1) and (2.26)-(2.27), one easily gets

$$\Phi(\mu)|_{G=SO(3)} = \tilde{R}(\tilde{\Gamma}(\mu))|_{G=SO(3)} = \frac{2(2\mu^3+7\mu^2+25\mu+20)}{(\mu^2+4)^2} \tag{7.21}$$

For large μ we obtain

$$\Phi(\mu)|_{G=SO(3)} = \tilde{R}(\tilde{\Gamma}(\mu))|_{G=SO(3)} \sim \frac{4}{\mu} \tag{7.22}$$

Let us consider the cubic equation

$$2\mu^3 + 7\mu^2 + 5\mu + 20 = 0 \tag{7.23}$$

and let us find its roots using the Cardano method. Equation (7.23) can be reduced to the standard form

$$y^3-\frac{19}{12}y+\frac{620}{27}=0 \tag{7.24a}$$

using the substitution

$$\mu=y-7/8 \tag{7.24b}$$

The resolvent equation for (7.23) looks like

$$z^2+\frac{620}{27}z-\frac{19^3}{27\cdot 12^3}=0 \tag{7.25}$$

The discriminant of (7.25) is

$$D=q^2+2\frac{4p^3}{27}=\frac{3583541}{27^3\cdot 4^2}=\frac{12\cdot 17\cdot 67\cdot 163}{24\cdot 3^9}>0 \tag{7.26}$$

Thus, the equation possesses only one real root,

$$y_1=-\frac{1}{3}\left\{\left[\frac{1}{24}\left(\frac{3583541}{3}\right)^{1/2}-310\right]^{1/3}+\left[310+\frac{1}{24}\left(\frac{3583541}{3}\right)^{1/2}\right]^{1/3}\right\} \tag{7.27}$$

For this

$$0>\mu_0=-\frac{1}{3}\left\{\frac{7}{2}+\left[\frac{1}{24}\left(\frac{3583541}{3}\right)^{1/2}-310\right]^{1/3}+\left[310+\frac{1}{24}\left(\frac{3583541}{3}\right)^{1/2}\right]^{1/3}\right\} \tag{7.28}$$

is the only real root of equation (7.23).

Thus, one gets

$$\tilde{R}(\tilde{\Gamma}(\mu_0))|_{G=SO(3)}=0 \tag{7.29}$$

and the cosmological constant vanishes. One calculates μ_0 and gets

$$\mu_0=-5.557669363\cdots \tag{7.30}$$

Moreover, note that the connection (7.20a)-(7.20b) does not satisfy (7.4). Thus, it should be rejected as unphysical, and because of this the $SO(3)$ group is not unphysical.

Let us consider (7.3) for dim $G>4$ and rewrite it for indices *abc*, *cab*, *bca*. One gets

$$\begin{aligned} l_{db}\tilde{\Gamma}^d{}_{ac}+l_{ad}\tilde{\Gamma}^d{}_{cb}&=-l_{bd}C^d{}_{ac}\\ l_{da}\tilde{\Gamma}^d{}_{ac}+l_{cd}\tilde{\Gamma}^d{}_{ba}&=-l_{ad}C^d{}_{cb}\\ l_{dc}\tilde{\Gamma}^d{}_{ba}+l_{bd}\tilde{\Gamma}^d{}_{cb}&=-l_{cd}C^d{}_{ba}\end{aligned} \tag{7.31}$$

Adding these equations, one gets

$$\tilde{\Gamma}_{bac}+\tilde{\Gamma}_{acb}+\tilde{\Gamma}_{cba}=-\tfrac{1}{2}(l_{bd}C^{d}{}_{ac}+l_{ad}C^{d}{}_{cb}+l_{cd}C^{d}{}_{ba}) \tag{7.32}$$

where $\tilde{\Gamma}_{bac}=h_{db}\tilde{\Gamma}^{d}{}_{ac}$. Substituting (7.12) into (7.32), one gets

$$C^{p}{}_{ac}(\Xi_{bp}-2\Xi_{pb})+C^{p}{}_{cb}(\Xi_{ap}-2\Xi_{pa})+C^{p}{}_{ba}(\Xi_{cp}-2\Xi_{pc})$$
$$=-\tfrac{1}{2}(l_{bd}C^{d}{}_{ac}+l_{ad}C^{d}{}_{cb}+l_{cd}C^{d}{}_{ba}) \tag{7.33}$$

Equation (7.33) is identically satisfied if

$$\Xi_{ab}-2\Xi_{ba}=-\tfrac{1}{2}l_{ab} \tag{7.34}$$

Thus, we get

$$\Xi_{ab}=\frac{1}{6}(2l_{ab}+l_{ba})=\frac{1}{2}h_{ab}-\frac{\mu}{6}k_{ab} \tag{7.35}$$

or

$$\Xi^{d}{}_{p}=\frac{1}{2}\delta^{d}{}_{p}-\frac{\mu}{6}k^{d}{}_{p} \tag{7.35a}$$

where $k^{d}{}_{p}=h^{de}k_{ep}$. Substituting (7.35a) into (7.12), one gets

$$\tilde{\Gamma}^{d}{}_{ef}=\frac{1}{2}C^{d}{}_{fe}+\Delta^{d}{}_{ef}=-\left[\frac{1}{2}C^{d}{}_{ef}+\frac{\mu}{6}(C^{p}{}_{ef}k^{d}{}_{p}-C^{d}{}_{pf}k^{p}{}_{e}+C^{d}{}_{pe}k^{p}{}_{f})\right] \tag{7.36}$$

Substituting the last equation into (7.2), one finds

$$\tilde{R}(\tilde{\Gamma})=-\frac{(4n+l^{ba}l_{ab})}{4(n-1)} \tag{7.37}$$

for $n>4$. For l_{ab} a right-invariant quantity, we get

$$\tilde{R}(\tilde{\Gamma}(g))=\tilde{R}(\tilde{\Gamma}(\varepsilon))=-\frac{(4n+\tilde{l}^{ba}(\varepsilon)\tilde{l}_{ab}(\varepsilon))}{4(n-1)} \tag{7.38}$$

where $\varepsilon\in G$ and it is a unit element of G.

Thus, we get, using (2.39)-(2.39a),

$$\tilde{R}(\tilde{\Gamma})=-\frac{(5n\Delta+2\sum_{j=1}^{[n/2]}\zeta^{j}(\Delta_{2j,2j-1}-\Delta_{2j-1,2j}))}{4\Delta(n-1)} \tag{7.39}$$

where $\Delta=\det(\tilde{l}_{ab}(\varepsilon))$ and $\Delta_{2j,2j-1}$, $\Delta_{2j-1,2j}$ are minors of $\tilde{l}_{ab}(\varepsilon)$. Thus, $\tilde{R}(\tilde{\Gamma})$ is a rational function of ζ^{j}, $j=1,2,\ldots,[n/2]$, and it is zero if the polynomial of ζ^{j} is

$$5n\Delta+2\sum_{j=1}^{[n/2]}\zeta^{j}_{c}(\Delta_{2j,2j-1}-\Delta_{2j-1,2j})=0 \tag{7.40}$$

for some set of ξ^{j}_{c}, $j=1,2,\ldots,[n/2]$.

Moreover, we can rewrite (7.39) in the more convenient form

$$\tilde{R}(\tilde{\Gamma}) = -\left(5n - 2\sum_{j=1}^{[n/2]} \frac{(-(\xi^j)^2 + \zeta^j \lambda_{2j-1})}{((\xi^j)^2 + \lambda_{2j}\lambda_{2j-1})}\right)[4(n-1)]^{-1} \tag{7.41}$$

In the case of $\lambda_i = \lambda$ the formula (7.41) is simplified and if simultaneously we put $\xi^j = \mu$, then we get

$$\tilde{R}(\tilde{\Gamma}) = -\frac{[\mu^2(5n - 2[n/2]) - 2\mu\lambda[n/2] + 5n\lambda^2]}{(\mu^2 + \lambda^2)} = \Phi(\mu) \tag{7.42}$$

Let us consider the quadratic equation

$$\mu^2\left(5n - 2\left[\frac{n}{2}\right]\right) - 2\mu\lambda\left[\frac{n}{2}\right] + 5n\lambda^2 = 0 \tag{7.43}$$

The discriminant of (7.43) is

$$D = \lambda^2\left(\left[\frac{n}{2}\right]^2 - 5n\left(5n + 2\left[\frac{n}{2}\right]^2\right)\right) \tag{7.44}$$

and

$$\left[\frac{n}{2}\right]^2 - 5n\left(5n - 2\left[\frac{n}{2}\right]\right) < 0 \tag{7.45}$$

Thus, we do not have real roots.

For large μ we get

$$\tilde{R}(\tilde{\Gamma}) \sim -\frac{(5n - 2[n/2])}{4(n-1)} \tag{7.46}$$

Let us consider the general case for $\tilde{R}(\tilde{\Gamma})$, i.e., we do not suppose equations (2.39)-(2.39a). Thus, in this case h_{ab} and k_{ab} do not commute. In order to do this, let us consider the matrix

$$L_b{}^n = l^{an}(\varepsilon) l_{ba}(\varepsilon) \tag{7.47}$$

Thus,

$$l^{ab}(\varepsilon) l_{ba}(\varepsilon) = \mathrm{Tr}(L_b{}^n) \tag{7.48}$$

In this way one gets

$$\mathrm{Tr}(L_b{}^n) = \sum_{i=1}^{n} \rho_i \tag{7.49}$$

where

$$L_b{}^n e^b(i) = \rho_i e^n(i) \tag{7.50}$$

In equation (7.49) we sum over all eigenvalues with their multiplicities. They can be complex. Moreover, the sum is real. Equation (7.50) can be transformed into

$$(\lambda_i h_{ab} + \mu k_{ab}(\varepsilon))\, e^b(i) = 0 \tag{7.51}$$

where

$$\lambda_i = \frac{\rho_i - 1}{\rho_i + 1}, \qquad \rho_i + 1 \neq 0 \tag{7.52}$$

or

$$\rho_i = \frac{1 + \lambda_i}{1 - \lambda_i}, \qquad \lambda_i \neq 1 \tag{7.53}$$

In this way we get

$$l^{ab}(\varepsilon) l_{ba}(\varepsilon) = \sum_{i=1}^{n} \frac{1 + \lambda_i}{1 - \lambda_i} \tag{7.54}$$

where we have

$$l(\lambda) = \det(\lambda h_{ab} + \mu k_{ab}(\varepsilon)) = 0 \tag{7.55}$$

In equation (7.54) we sum over all roots of (7.55) with their multiplicities. Thus, we have

$$\tilde{R}(\tilde{\Gamma}) = -\left(4n + \sum_{i=1}^{n} \frac{1 + \lambda_i}{1 - \lambda_i}\right)[4(n-1)]^{-1} \tag{7.56}$$

We can rewrite (7.56) in the form

$$\tilde{R}(\tilde{\Gamma}) = -\left(4n + \sum_{i=1}^{n} \left(\frac{1 + \mu\zeta_i}{1 - \mu\zeta_i}\right)\right)[4(n-1)]^{-1} \tag{7.57}$$

where we have

$$\det(\zeta h_{ab} + k_{ab}) = 0 \tag{7.58}$$

and μ is such that $1 - \mu\zeta_i \neq 0$ for any ζ_i satisfying equation (7.58). Thus, we have for the cosmological constant

$$\Phi(\mu) = -\left\{4n \prod_{i=1}^{n} (1 - \mu\zeta_i) + \sum_{i=1}^{n} (1 + \mu\zeta_i) \prod_{\substack{j \neq i \\ j=1}}^{n} (1 - \mu\zeta_j)\right\} \Bigg/ \left\{4(n-1) \prod_{i=1}^{n} (1 - \mu\zeta_i)\right\} \tag{7.59}$$

It is easy to see that $\Phi(\mu)=0$ if

$$4n\prod_{i=1}^{n}(1-\mu\zeta_i)+\sum_{i=1}^{n}(1+\mu\zeta_i)\prod_{\substack{j\neq i\\ j=1}}^{n}(1-\mu\zeta_j)=0 \tag{7.60}$$

which is an algebraic equation of nth order.

Let us suppose that $\zeta_i=\zeta$ for $i=1,2,\ldots,n$. In this case one easily gets

$$\mu_0=\frac{5}{3\zeta} \tag{7.61}$$

and $\Phi(\mu_0)=0$.

8. CONFORMAL TRANSFORMATION OF $g_{\mu\nu}$. TRANSFORMATION OF THE SCALAR FIELD ρ

In Section 6 we obtained the Lagrangian density $B(W)$,

$$\frac{1}{|l|^{1/2}}B(W)=(-g)^{1/2}\rho^n\bar{R}(W)+\rho^{n-2}L_{\rm scal}(\rho)+8\pi\rho^{n+2}L_{\rm YM}$$
$$+(-g)^{1/2}\tilde{R}(\tilde{\Gamma})\rho^{n-2} \tag{8.1}$$

where

$$L_{\rm YM}=-\frac{(-g)^{1/2}}{8\pi}(2l_{cd}H^cH^d-l_{cd}g^{\alpha\omega}g^{\beta\mu}L^c{}_{\alpha\beta}H^d{}_{\omega\mu}) \tag{8.2}$$

$$L_{\rm scal}=(-g)^{1/2}(m\tilde{g}^{(\gamma\nu)}+n^2g^{[\mu\nu]}g_{\delta\mu}\tilde{g}^{(\delta\gamma)})\rho_{,\nu}\rho_{,\gamma} \tag{8.3}$$

$L_{\rm YM}$ is the Lagrangian density for the Yang–Mills field and $L_{\rm scal}(\rho)$ is the Lagrangian density for the scalar field ρ. This Lagrangian $[B(W)]$ is in the same form as in Bergmann's paper (see ref. 50). Bergmann considers the general Lagrangian for the tensor–scalar theory of gravitation, including Jordan–Thiry theory and Brans–Dicke theory (refs. 93 and 94). In his Lagrangian there are four arbitrary functions of the scalar field, f_1, f_2, f_3, f_4. In our case we have

$$\begin{aligned} f_1(\rho)&=\rho^n\\ f_2(\rho)&=8\pi\rho^{n+2}\\ f_3(\rho)&=\rho^{n+2}\\ f_4(\rho)&=\rho^{n-2}\end{aligned} \tag{8.4}$$

There are of course some differences, since our theory is nonsymmetric.

There exists a skew-symmetric part of $g_{\mu\nu}$, and because of this, we have a different Lagrangian for the scalar field ρ. Simultaneously, in the place of the ordinary Ricci scalar of curvature we have the Moffat–Ricci scalar of curvature. In the place of the Lagrangian for the Maxwell field we have a Lagrangian for the Yang–Mills field in the nonsymmetric version. Moreover, the general features are the same. We really get a scalar–tensor (nonsymmetric) theory. Now we proceed with the conformal transformation for the metric $g_{\mu\nu}$ and the transformation of the scalar field ρ. This is only the redefinition of $g_{\mu\nu}$ and ρ,

$$\rho = e^{-\Psi} \tag{8.5}$$

$$g_{\mu\nu} \to e^{n\Psi} g_{\mu\nu} = \frac{1}{\rho^n} g_{\mu\nu} \tag{8.6}$$

This procedure is of course from ref. 50. The only difference is that $g_{\mu\nu}$ is now nonsymmetric. After the transformations (8.5) and (8.6) we get

$$\frac{1}{|l|^{1/2}} B(W) = (-g)^{1/2}\{\bar{R}(\bar{W}) + e^{(2+n)\Psi}\tilde{R}(\tilde{\Gamma}) + 8\pi\, e^{-(n+2)\Psi}\mathscr{L}_{\mathrm{YM}} + \mathscr{L}_{\mathrm{scal}}(\Psi)\} \tag{8.7}$$

where

$$\begin{aligned} \mathscr{L}_{\mathrm{scal}}(\Psi) &= (m\tilde{g}^{(\gamma\nu)} + n^2 g^{[\mu\nu]} g_{\delta\mu}\tilde{g}^{(\delta\gamma)})\Psi_{,\nu}\Psi_{,\gamma} \\ m &= l^{[dc]} l_{[dc]} - 3n(n-1) \end{aligned} \tag{8.8}$$

It is easy to see that the scalar field Ψ is chargeless (it has no color changes). However, it couples to the gauge (Yang–Mills) field due to the term

$$8\pi\, e^{-(n+2)\Psi}\mathscr{L}_{\mathrm{YM}} \tag{8.9}$$

It couples also to the cosmological constant

$$e^{(n+2)\Psi}\tilde{R}(\tilde{\Gamma}) \tag{8.10}$$

These two terms, (8.9) and (8.10), suggest that the scalar field is massive. This is different than in Brans–Dicke theory, where the scalar field couples to the trace of the energy-momentum tensor for matter (refs. 93 and 94).

We can consider a more general expression, i.e., $[R(W)+\beta]\sqrt{\gamma}$, where β is a constant. In this case we get one more term in the expression (8.1), i.e., $(|l|^{1/2}\beta(-g)^{1/2})\rho^n$ or $(|l|^{1/2}\beta(-g)^{1/2})\, e^{n\Psi}$. This term plays the role of an additional cosmological term and can be added to the term Φ, leading to

$$\Phi_{\mathrm{eff}} = e^{n\Psi}(\beta + e^{2\Psi}\tilde{R}(\tilde{\Gamma})$$

For $\rho = 1$ ($\Psi = 0$) this leads to the effective cosmological constant $\beta + \tilde{R}(\tilde{\Gamma})$. Taking $\beta = -\tilde{R}(\tilde{\Gamma})$, we can make it vanish. However, for ρ (or Ψ) nonconstant, this does not work. Fortunately we have a completely different mechanism of vanishing of the cosmological term and we do not need such manipulations. Moreover, some authors use this additional term in the classical non-Abelian Kaluza–Klein theory in order to solve the problem of the cosmological constant. We do not consider this to be in the spirit of the original Kaluza–Klein theory and no longer consider this additional term.

9. GAUGE INVARIANCE OF THE LAGRANGIAN

Let us consider the problem of gauge invariance of the Lagrangian in our theory. In geometrical language this means that

$$\Phi'(g)\mathscr{L}=\mathscr{L} \qquad \text{or} \qquad X_a\mathscr{L}=0 \tag{9.1}$$

where X_a means the infinitesimal right-action of the group G on $\underline{P}$ in the direction a. Equation (9.1) is satisfied, because of the right-invariance of the connection ω_{AB}. Moreover, we will check it here independently, starting from equation (6.33), which defines the Lagrangian $\mathscr{L}$. Equation (9.1) is equivalent to

$$\hat{\nabla}_a\mathscr{L}=0 \tag{9.2}$$

[see (3.27) for the definition of $\hat{\nabla}_a$]. One easily checks that

$$\hat{\nabla}_a\tilde{R}(\tilde{\Gamma})=0=\hat{\nabla}_a m \tag{9.3}$$

In the same way we can check that

$$\hat{\nabla}_a\mathscr{L}_{\mathrm{YM}}=0 \tag{9.4}$$

supposing that $L^a{}_{\mu\nu}$ is an Ad quantity (Ad type). Moreover, we are supposed to check the invariance of the formula (4.10) which defines $L^a{}_{\mu\nu}$.

Let us rewrite (4.10) in the following form:

$$Q^e{}_d g_{\delta\beta}g^{\sigma\delta}L^d{}_{\sigma\alpha}+g_{\alpha\delta}g^{\delta\sigma}L^e{}_{\beta\sigma}=2g_{\alpha\delta}g^{\delta\sigma}H^e{}_{\beta\sigma} \tag{9.5}$$

where

$$Q^e{}_d=l^{ce}l_{dc} \tag{9.6}$$

Let us act on both sides of (9.5) by X_f. We get

$$\begin{aligned}X_fQ^e{}_d(g_{\delta\beta}g^{\sigma\delta}L^d{}_{\sigma\alpha})+Q^e{}_dD^d{}_{qf}(g_{\delta\beta}g^{\sigma\delta}L^q{}_{\sigma\alpha})\\+g_{\alpha\delta}g^{\delta\sigma}D^e{}_{df}L^d{}_{\beta\sigma}=2g_{\alpha\delta}g^{\delta\sigma}C^e{}_{df}H^d{}_{\beta\sigma}\end{aligned} \tag{9.7}$$

where we put

$$\begin{aligned}X_fL^d{}_{\sigma\alpha}&=D^d{}_{qf}L^q{}_{\sigma\alpha}\\X_fH^d{}_{\sigma\alpha}&=C^d{}_{qf}H^q{}_{\sigma\alpha}\end{aligned} \tag{9.8}$$

The last condition means that $H^q{}_{\sigma\alpha}$ is an Ad quantity. $D^d{}_{qf}$ defines a transformation property of $L^a{}_{\mu\nu}$.

Using equation (9.5), one easily gets

$$g_{\delta\beta}g^{\sigma\delta}L^d{}_{\sigma\alpha}(X_f Q^e{}_d + Q^e{}_q D^q{}_{df} - C^e{}_{pf}Q^p{}_d) + g_{\alpha\delta}g^{\delta\sigma}L^d{}_{\beta\sigma}(D^e{}_{df} - C^e{}_{df}) = 0 \tag{9.9}$$

Equation (9.9) must be identically satisfied for every $L^d{}_{\sigma\alpha}$ and $g_{\alpha\beta}$. Thus, we get

$$D^e{}_{df} = C^3{}_{df} \tag{9.10}$$

$$X_f Q^e{}_d + Q^e{}_q D^q{}_{df} - C^e{}_{pf}Q^p{}_d = 0 \tag{9.11}$$

Equation (9.10) means that $L^d{}_{\mu\nu}$ is an Ad quantity and equation (9.11) is equivalent to

$$\hat{\nabla}_f Q^e{}_d = 0 \tag{9.12}$$

This is equivalent to

$$\hat{\nabla}_f l_{ab} = 0 \tag{9.12a}$$

which is the condition for l_{ab} known from Section 2.

Now the gauge invariance of $\mathscr{L}_{\rm YM}$ is easily satisfied.

Let us prove the gauge invariance of the Yang-Mills Lagrangian $\mathscr{L}_{\rm YM}$ in a different way, supposing that l_{cd} is right-invariant and $L^{c\mu\nu}$ is of Ad type. One has

$$\mathscr{L}_{\rm YM} = \frac{1}{8\pi} l_{cd}(H^c H^d - L^{c\mu\nu}H^d{}_{\mu\nu}) \tag{9.13}$$

Let us take two different gauges (two local sections e and f of the bundle $\underline{P}$) and

$$\begin{aligned} e^*L^{c\mu\nu}(e(x)) &= B^{c\mu\nu}(x) \\ e^*H^d{}_{\mu\nu}(e(x)) &= F^d{}_{\mu\nu}(x) \end{aligned} \tag{9.14}$$

$$\begin{aligned} f^*L^{c\mu\nu}(f(x)) &= \bar{B}^{c\mu\nu}(x) \\ f^*H^d{}_{\mu\nu}(f(x)) &= \bar{F}^d{}_{\mu\nu}(x) \end{aligned} \tag{9.15}$$

$$\begin{aligned} e^*l_{cd}(e(x)) &= m_{cd}(x) \\ f^*l_{cd}(f(x)) &= \bar{m}_{cd}(x) \end{aligned} \tag{9.16}$$

One writes $m_{ab}(x) = h_{ab} + \mu n_{ab}(x)$, where $n_{ab}(x) = e^*k_{ab}(e(x))$. Let the matrix $U = (U^d{}_a) = \mathrm{Ad}_{g^{-1}(x)}$ be a gauge-changing matrix. One gets

$$\begin{aligned} \bar{F}^d{}_{\mu\nu} &= U^d{}_a F^a{}_{\mu\nu} \\ \bar{B}^{d\mu\nu} &= U^d{}_a B^{a\mu\nu} \\ \bar{m}_{cd} &= \bar{U}^k{}_c \bar{U}^l{}_d m_{kl} \end{aligned} \tag{9.17}$$

i.e., $(\bar{U}^k{}_d)=\bar{U}=\mathrm{Ad}_{g(x)}$, where

$$\bar{U}^k{}_c U^c{}_d=\delta^k{}_d$$

One easily gets

$$\mathscr{L}_{\mathrm{YM}}(e(x))=\mathscr{L}_{\mathrm{YM}}(f(x)) \tag{9.18}$$

which proves the gauge invariance of the Lagrangian.

This result can be obtained directly using a technique of a vertical projection of any curve $t\to p(t)\in \underline{P}$. In this way we prove that the $\mathscr{L}_{\mathrm{YM}}(p(t))$ does not depend on a gauge on any curve in $\underline{P}$. Let us consider a gauge $e\to r(e)$ and define a gauge-dependent vertical projection $t\to g(t)\in G$ of the curve $t\to p(t)$ using the formula

$$p(t)=r(e(t))g(t) \tag{9.19}$$

We get

$$H^a{}_{\mu\nu}(p(t))=U^a{}_{a'}(g(t)^{-1})F^{a'}{}_{\mu\nu}(e(t)) \tag{9.20}$$

$$L^d{}_{\mu\nu}(p(t))=U^d{}_{a'}(g(t)^{-1})B^{a'}{}_{\mu\nu}(e(t)) \tag{9.21}$$

$$l_{ad}(p(t))=U^{a'}{}_a(g(t))U^{d'}{}_d(g(t))m_{a'd'}(e(t)) \tag{9.22}$$

Thus, for any gauge e and for any curve $t\to p(t)$, one gets

$$l_{ad}(p(t))H^a{}_{\mu\nu}(p(t))L^{d\mu\nu}(p(t))=m_{ad}(e(t))F^a{}_{\mu\nu}(e(t))B^{d\mu\nu}(e(t)) \tag{9.23}$$

i.e.,

$$\mathscr{L}_{\mathrm{YM}}(p(t))=\mathscr{L}_{\mathrm{YM}}(e(t)) \tag{9.24}$$

We can also work in a different way using properties of $H^a{}_{\mu\nu}$, $L^a{}_{\mu\nu}$, and l_{cd} with respect to the right action of the group G on $\underline{P}$. We get

$$\begin{aligned}\Phi'(g)\mathscr{L}_{\mathrm{YM}}&=\Phi'(g)(l_{cd}H^c{}_{\mu\nu}L^{d\mu\nu})\\&=(\Phi'(g)l_{cd})(\varphi'(g)H^c{}_{\mu\nu})(\varphi'(g)L^{d\mu\nu})\\&=l_{c'd'}U^{c'}{}_c(g)U^{d'}{}_d(g)U^c{}_a(g^{-1})H^a{}_{\mu\nu}U^d{}_b(g^{-1})L^{b\mu\nu}\\&=l_{c'd'}H^c{}_{\mu\nu}L^{d'\mu\nu}=\mathscr{L}_{\mathrm{YM}}\end{aligned} \tag{9.25}$$

It is worth noticing that in general $\nabla^{\mathrm{gauge}}_\mu l_{ab}\neq 0$; ∇^{gauge} means a gauge derivative with respect to the connection ω.

10. VARIATIONAL PRINCIPLE. EQUATIONS OF FIELDS. INTERPRETATION AND CONCLUSIONS

Let us consider the Palatini variational principle for $R(W)$,

$$\begin{gathered}\delta\int_V \gamma^{1/2}R(W)\,d^{(n+4)}x=0\\ V\subset \underline{P},\qquad v=U\times G,\quad U\subset E\end{gathered} \tag{10.1}$$

$d^{(n+4)}x = d^4x\, d\mu_G(g)$, where $d\mu_G(g)$ is a bi-invariant measure on a group G (identified with a typical fiber via $\varphi_x : G \to F_x \simeq G$, $x \in U$). It is easy to see that (10.1) is equivalent to [see (6.33)]

$$\frac{1}{|l|^{1/2}}\,\delta \int_U B(W)(-g)^{1/2}\, d^4x = 0, \qquad U \subset E \tag{10.2}$$

Using the gauge invariance of the scalar curvature, we integrate over a group G.

We redefine $g_{\mu\nu}$ and ρ as in Section 8. Thus, we have the following independent quantities: $g_{\mu\nu}$, $\bar{W}^\lambda{}_{\mu\nu}$, Ψ, and ω. We vary with respect to the independent quantities. After some calculations we get

$$\bar{R}_{\mu\nu}(\bar{W}) - \tfrac{1}{2} g_{\mu\nu}\bar{E}(\bar{W}) = 8\pi K\big(\overset{\text{gauge}}{T_{\mu\nu}} + \overset{\text{scal}}{T_{\mu\nu}}(\Psi) + g_{\mu\nu}\varphi\big) \tag{10.3}$$

$$\underset{\sim}{g}^{[\mu\nu]}{}_{,\nu} = 0 \tag{10.4}$$

$$g_{\mu\nu,\sigma} - g_{\zeta\nu}\bar{\Gamma}^\zeta{}_{\mu\sigma} - g_{\mu\zeta}\bar{\Gamma}^\zeta{}_{\sigma\nu} = 0 \tag{10.5}$$

$$\begin{aligned}\overset{\text{gauge}}{\nabla_\mu}\,(l_{ab}\underset{\sim}{L}^{a\alpha\mu}) &= 2\underset{\sim}{g}^{[\alpha\beta]}\,\overset{\text{gauge}}{\nabla_\beta}\,(h_{ab}\underset{\sim}{g}^{[\mu\nu]}H^a{}_{\mu\nu}) \\ &\quad - (n+2)\partial_\beta\Psi[l_{ab}L^{a\beta\alpha} - 2g^{[\beta\alpha]}(h_{ab}g^{[\mu\nu]}H^a{}_{\mu\nu})]\end{aligned} \tag{10.6}$$

$$\begin{aligned}&[(n^2+2m)\tilde{g}^{(\alpha\mu)} - n^2 g^{\nu\mu}\tilde{g}^{(\alpha\sigma)}]\frac{\partial^2\Psi}{\partial x^\alpha\,\partial x^\mu} + \frac{1}{(-g)^{1/2}}\partial_\mu \\ &\times\left\{(-g)^{1/2}\left[n^2\tilde{g}^{(\mu\alpha)} - \frac{n^2}{2}g_{\delta\nu}(g^{\nu\alpha}\tilde{g}^{(\mu\delta)} + g^{\nu\mu}\tilde{g}^{(\alpha\delta)}) - 2m\tilde{g}^{(\mu\alpha)}\right]\right\}\frac{\partial\Psi}{\partial x^\alpha} \\ &- 8(n+2)\pi\, e^{-(n+2)\Psi}(\mathscr{L}_{\text{YM}} - 2\Phi) = 0\end{aligned} \tag{10.7}$$

where

$$\begin{aligned}\overset{\text{gauge}}{T_{\alpha\beta}} = -\frac{l_{ab}}{4\pi}\Big\{ & g_{\gamma\beta}g^{\tau\zeta}g^{\varepsilon\gamma}L^a{}_{\zeta\alpha}L^b{}_{\tau\varepsilon} - 2g^{[\mu\nu]}H^{(a}{}_{\mu\nu}H^{b)}{}_{\alpha\beta} \\ & - \frac{1}{4}g_{\alpha\beta}[L^{a\mu\nu}H^b{}_{\mu\nu} - 2(g^{[\mu\nu]}H^a{}_{\mu\nu})(g^{[\gamma\sigma]}H^b{}_{\gamma\sigma})]\Big\}\end{aligned} \tag{10.8}$$

is the energy-momentum tensor for the gauge (Yang–Mills) field, and

$$\begin{aligned}\overset{\text{scal}}{T_{\alpha\beta}}(\Omega) = \frac{e^{(n+2)\Psi}}{16\pi}\Big\{ & (g_{\varkappa\alpha}g_{\omega\beta} + g_{\omega\alpha}g_{\varkappa\beta})\tilde{g}^{(\gamma\varkappa)}\tilde{g}^{(\nu\omega)} \\ & \times\left[\frac{n^2}{2}(g^{\zeta\mu}g_{\nu\zeta} - \delta^\mu{}_\nu)\Omega_{,\mu} + m\Psi_{,\nu}\right]\Psi_{,\gamma} \\ & - g_{\alpha\beta}(m\tilde{g}^{(\gamma\nu)} + n^2 g^{[\mu\nu]}g_{\delta\mu}\tilde{g}^{(\gamma\delta)})\Psi_{,\nu}\Psi_{,\gamma}\Big\}\end{aligned} \tag{10.9}$$

is the energy-momentum tensor for the scalar field Ψ. We have

$$K = e^{-(n+2)\Psi} \tag{10.10}$$

It plays the role of a gravitational "constant"

$$\Phi = e^{2(n+2)\Psi} \frac{\tilde{R}(\Gamma)}{16\pi} \tag{10.11}$$

Φ plays the role of a cosmological "constant" (cosmological term)

$$m = (l^{[dc]} l_{[dc]} - 3n(n-1)) \tag{10.12}$$

$$\begin{aligned} \underset{\sim}{L}^{a\mu\nu} &= (-g)^{1/2} g^{\beta\mu} g^{\gamma\alpha} L^{a}{}_{\beta\gamma} \\ \underset{\sim}{g}^{[\mu\nu]} &= (-g)^{1/2} g^{[\mu\nu]} \end{aligned} \tag{10.13}$$

$\nabla^{\text{gauge}}_{\mu}$ means gauge derivative and

$$l_{dc} g_{\mu\beta} g^{\gamma\mu} L^{d}{}_{\gamma\alpha} + l_{cd} g_{\alpha\mu} g^{\mu\gamma} L^{d}{}_{\beta\gamma} = 2 l_{cd} g_{\alpha\mu} g^{\mu\gamma} H^{d}{}_{\beta\gamma} \tag{10.14}$$

Equation (10.14) can be rewritten in a matrix notation

$$g(g^{-1})^{T}(l * \mathbf{L}) + g^{T} g^{-1}(l^{T} * \mathbf{L}^{T}) = 2 g^{T} g^{-1}(l^{T} * \mathbf{H}^{T}) \tag{10.14a}$$

where T means matrix transposition and "$*$" means the action of an $(n \times n)$ matrix on an n-dimensional vector.

The left-hand side of equation (10.6) can be rewritten as $(-g)^{1/2}$ $\bar{\nabla}^{\text{gauge}}_{\mu}(l_{ab} L^{a\alpha\mu})$, where $\bar{\nabla}^{\text{gauge}}_{\mu}$ means the covariant derivative with respect to the connection $\bar{\omega}^{\alpha}{}_{\beta}$ on E and "gauge" at once.

Equations (10.3) and (10.4) are equations from NGT with Yang-Mills and scalar sources with a cosmological constant. Both constants, gravitational and cosmological, depend here on the scalar field Ψ, which propagates according to (10.7). Equation (10.5) is the compatibility condition for the connection $\bar{\omega}^{\alpha}{}_{\beta}(\bar{\Gamma}^{\alpha}{}_{\beta\gamma})$. Equation (10.6) is the second Yang-Mills equation with sources for $L^{a\mu\nu}$. The first Yang-Mills equation is of course the Bianchi identity for the connection ω [see equation (1.8)]. It is easy to see that

$$g^{\alpha\beta} \overset{\text{gauge}}{T}_{\alpha\beta} = 0 \tag{10.15}$$

$$g^{\alpha\beta} \overset{\text{scal}}{T}_{\alpha\beta}(\Psi) \neq 0 \tag{10.16}$$

Now we are able to interpret all quantities in our theory. First of all, it is easy to see that $L^{a}{}_{\alpha\beta}$ plays the role of the second tensor of the Yang-Mills field (gauge) (i.e., an induction tensor) strength and equation (10.11) expresses the relationship between both tensors $H^{a}{}_{\alpha\beta}$ and $L^{a}{}_{\alpha\beta}$.

In the electromagnetic case $[G=U(1)]$, we have to deal with tensors $F_{\alpha\beta}$ and $H_{\alpha\beta}$ (see refs. 18 and 19), which are the first and second tensors of the electromagnetic strength (an ordinary and an induction one). In the classical electrodynamics of continuous media (ref. 68) or in nonlinear electrodynamics (ref. 69), it is necessary to define both of these tensors. The first tensor $F_{\alpha\beta}$ is built from $(\mathbf{E},\mathbf{B})$ and the second from $(\mathbf{D},\mathbf{H})$. Here we build $H^a{}_{\alpha\beta}$ from $(\mathbf{E}^a,\mathbf{B}^a)$ and $L^a{}_{\alpha\beta}$ from $(\mathbf{D}^a,\mathbf{H}^a)$. For example, in quantum chromodynamics we have to deal with $\mathbf{D}^a$ (see ref. 95). The vacuum behaves as a dielectric for the gluon field.

If the metrics $g_{\alpha\beta}$ and l_{ab} are symmetric, $H^a{}_{\alpha\beta}=L^a{}_{\alpha\beta}$. Thus, the skew-symmetric part of the metric γ_{AB} induces Yang-Mills polarization of the vacuum (see refs. 23 and 24).

In the electromagnetic case (see refs. 18, 19, and 25) $[G=U(1)]$, we define the electromagnetic polarization tensor of the vacuum $M_{\alpha\beta}$ induced by the skew-symmetric part of the metric such that

$$H_{\alpha\beta}=F_{\alpha\beta}-4\pi M_{\alpha\beta} \tag{10.17}$$

($L^a{}_{\alpha\beta}$ is analogous to $H_{\alpha\beta}$ and $H^a{}_{\alpha\beta}$ to $F_{\alpha\beta}$).

In the classical electrodynamics of continuous media (see ref. 68) or in nonlinear electrodynamics (see ref. 69) this tensor is usually defined.

Here we can define the tensor $M^a{}_{\alpha\beta}$ such that

$$L^a{}_{\alpha\beta}=H^a{}_{\alpha\beta}-4\pi M^a{}_{\alpha\beta} \tag{10.18}$$

$M^a{}_{\alpha\beta}$ is the Yang-Mills field analogue of the electromagnetic polarization tensor $M_{\alpha\beta}$. It is easy to see that

$$4\pi M^a{}_{\alpha\beta}==-K^a{}_{\alpha\beta} \tag{10.19}$$

[see equation (6.13)]. Thus, we get a geometrical interpretation of $M^\alpha{}_{\alpha\beta}$,

$$Q^a{}_{\alpha\beta}(\Gamma)=8\pi M^a{}_{\alpha\beta} \tag{10.20}$$

($M^a{}_{\alpha\beta}$ is of course the Ad-type tensor defined on $\underline{P}$). Thus, the Yang-Mills field polarization induced by the skew-symmetric part of the metric γ_{AB} is the torsion in the additional dimensions. This is in very good agreement with the results from ref. 16. The only difference is that in ref. 16 the Yang-Mills field polarization has its origin from external sources and (10.20) plays the role of the Cartan equation in Kaluza-Klein theory with torsion. The skew-symmetric part of the metric γ_{AB} also changes the Yang-Mills field Lagrangian,

$$\mathscr{L}_{\rm YM}=-\frac{l_{ab}}{8\pi}[2(g^{[\alpha\beta]}H^a{}_{\alpha\beta})(g^{[\mu\nu]}H^b{}_{\mu\nu})-L^{a\mu\nu}H^b{}_{\mu\nu}] \tag{10.21}$$

For (10.21) we have a new term

$$-2h_{ab}(g^{[\alpha\beta]}H^a{}_{\alpha\beta})(g^{[\mu\nu]}H^b{}_{\mu\nu})$$

which is an interaction between the skewon field and the Yang–Mills field. This term vanishes if the metric of space-time is symmetric and is always nonnegative if the group G is compact. The second term in (10.21) is also different than in the classical Yang–Mills field Lagrangian. In place of the symmetric tensor h_{ab} we have now a nonsymmetric tensor

$$l_{ab} = h_{ab} + \mu k_{ab}$$

The skew-symmetric part of the metric induces also a source for the Yang–Mills field. In (10.6) we get a current

$$\begin{aligned} \underset{\sim}{J}^{\alpha}{}_{a} &= \frac{1}{2\pi} \underset{\sim}{g}^{[\alpha\beta]} \overset{\text{gauge}}{\bar{\nabla}} (h_{ab} g^{[\mu\nu]} H^{b}{}_{\mu\nu}) \\ &\quad - \frac{n+2}{4\pi} \partial_\beta \Psi [l_{ab} \underset{\sim}{L}^{b\beta\alpha} - 2 \underset{\sim}{g}^{[\beta\alpha]} (h_{ab} g^{[\mu\nu]} H^{b}{}_{\mu\nu})] \\ &= \overset{(1)}{\underset{\sim}{J}^{\alpha}{}_{a}} + \overset{(2)}{\underset{\sim}{J}^{\alpha}{}_{a}}(\Psi) \end{aligned} \tag{10.22}$$

The current $\overset{(1)}{\underset{\sim}{J}^{\alpha}{}_{a}}$ vanishes if the metric is symmetric. All of the above new effects, from the nonsymmetric non-Abelian Jordan–Thiry theory concerning the Yang–Mills field, are also obtained in the nonsymmetric non-Abelian Kaluza–Klein theory (see ref. 23). If we put $\rho = 1$ or $\Psi = 0$ we get the nonsymmetric non-Abelian Kaluza–Klein theory (see ref. 23). Now we pass to new effects which appear because of the scalar field ρ (or Ψ). The scalar field propagates according to equation (10.7). This equation is more of the Klein–Gordon type than the wavetype. We have here a term

$$\begin{aligned} &-8(n+2)\pi\, e^{-(n+2)\Psi} (\mathscr{L}_{\text{YM}} - 2\Phi) \\ &\quad = -8(n+2)\pi (e^{-(n+2)\Psi} \mathscr{L}_{\text{YM}} - e^{(n+2)\Psi} \tilde{R}(\tilde{\Gamma})) \end{aligned} \tag{10.23}$$

For the electromagnetic case $[G = U(1)]$ we have only one term,

$$-24\pi\, e^{-3\Psi} \mathscr{L}_{\text{em}} \tag{10.24}$$

(see ref. 19). From observational data we know that the cosmological constant is very small (almost zero) and therefore this second term does not play any important role.

We know that

$$\tilde{R}(\tilde{\Gamma}) \sim \frac{\text{const}}{\mu} \quad \text{or} \quad \sim \text{const} \tag{10.25}$$

for very large μ and for this we can sometimes choose [for example, for $G = SO(3)$]

$$|\mu| \geq 10^{127} \tag{10.26}$$

Thus, we really have only the first term

$$-8(n+2)\pi\, e^{-(n+2)\Psi}\mathscr{L}_{\mathrm{YM}} \tag{10.27}$$

From observational data we know that the gravitational constant is almost constant. This means that $\rho \approx 1$ or $\Psi \approx 0$. Thus, we can expand $e^{-(n+2)\Psi} = 1-(n+2)\Psi+\cdots$ in (10.27) and leave only the first two terms

$$-8(n+2)\pi\, e^{-(n+2)\Psi}\mathscr{L}_{\mathrm{YM}} \approx -8(n+2)\pi\mathscr{L}_{\mathrm{YM}} + 8(n+2)^2\pi\mathscr{L}_{\mathrm{YM}}\Psi \tag{10.28}$$

The term $(8(n+2)^2\pi\mathscr{L}_{\mathrm{YM}})\Psi$ plays the role of the mass term for the scalar field Ψ in equation (10.7). Thus, it seems that the scalar field Ψ is massive. This statement is also supported by equation (10.16); the trace of the energy-momentum tensor for Ψ is not zero. Let us turn to the Lagrangian for the scalar field Ψ:

$$\mathscr{L}_{\mathrm{scal}}(\Psi) = (mg^{(\gamma\nu)} + n^2 g^{[\mu\nu]} g_{\delta\mu}\tilde{g}^{(\gamma\delta)})\Psi_{,\nu}\Psi_{,\gamma} \tag{10.29}$$

where $m = (l^{[dc]}l_{[dc]} - 3n(n-1))$ and $n = \dim G$.

It is easy to see that in the electromagnetic case $[G = U(1)]$ we get $m = 0$ and

$$\mathscr{L}_{\mathrm{scal}}(\Psi) = g^{[\mu\nu]} g_{\delta\mu}\tilde{g}^{(\gamma\delta)}\Psi_{,\nu}\Psi_{,\gamma} \tag{10.30}$$

i.e., the same formula as in ref. 19. In this case if the skew-symmetric part of the metric $g_{\mu\nu}$ is zero, then the scalar field Ψ does not propagate,

$$\mathscr{L}_{\mathrm{scal}}(\Psi) \equiv 0 \tag{10.31}$$

In higher-dimensional theory (higher than 5, $n \geq 2$) we get

$$\mathscr{L}_{\mathrm{scal}}(\Psi) = -3n(n-1)\tilde{g}^{(\gamma\nu)}\Psi_{,\nu}\Psi_{,\gamma} \neq 0 \tag{10.32}$$

Thus, we have propagation of the scalar field Ψ in the symmetric non-Abelian Jordan–Thiry theory. In the nonsymmetric case with $l_{ab} \neq l_{ba}$ we get the same feature as in the electromagnetic case if

$$m = 0 \tag{10.33}$$

This means that

$$l^{[dc]}l_{[dc]} = 3n(n-1) \tag{10.34}$$

In this case the scalar field Ψ does not propagate if the skew-symmetric part of $g_{\mu\nu}$ is zero.

Summing up, we get the following statement: the scalar field Ψ is probably massive. This has many important consequences. First of all Ψ is of short range and has Yukawa-type behavior

$$\Psi \sim \frac{1}{r}\, e^{-\alpha r} \tag{10.35}$$

Thus, Ψ does not violate the weak equivalence principle. The scalar force is of a short range. Thus, in our theory scalar forces connected to the gravitational constant ($K = e^{-(n+2)\Psi}$) do not violate the universal fall of all bodies. Due to the Yukawa-type behavior of Ψ, we get that, at long distances, the gravitational constant K is really constant.

Concluding, we see that the nonsymmetric Jordan–Thiry theory, combining Moffat's theory and Yang–Mills theory with the scalar field, is stronger than the classical Jordan–Thiry approach (or the Kaluza–Klein approach) combining general relativity and a gauge theory.

In the nonsymmetric non-Abelian Jordan–Thiry theory there exist "interference effects" between the gravitational and gauge fields which are absent in the classical approach (neglecting the appearance of the cosmological constant, which is a disadvantage of the theory, though it is possible to remove it in some approaches; see ref. 10). In the theory we get the following interference effects.

1. A new term in the Yang–Mills Lagrangian

$$-\frac{1}{4\pi} l_{ab}(g^{[\mu\nu]}H^a{}_{\mu\nu})(g^{[\alpha\beta]}H^b{}_{\alpha\beta})$$

2. A change in the classical part of the Yang–Mills field Lagrangian in replacing h_{ab} by l_{ab}.

3. The existence of the Yang–Mills field polarization of the vacuum $M^a{}_{\alpha\beta}$, which has a geometrical interpretation as a torsion in additional dimensions.

4. An additional term in the Kerner–Wong equation (the equation of the motion for a test particle in the gravitational and Yang–Mills fields)

$$\frac{1}{2}\left(\frac{q^b}{m_0}\right)(l_{bd}g^{\alpha\beta} - l_{db}g^{\beta\alpha})L^d{}_{\beta\gamma}u^\gamma$$

where m_0 is the rest mass of a test particle and q^b is its color (isotopic) charge.

5. A new energy-momentum tensor $T^{\text{gauge}}_{\alpha\beta}$ with zero trace.

6. Sources for the Yang–Mills field, the current $\underset{\sim}{J}^{\alpha a}$.

7. The existence of a scalar field Ψ (or ρ) with an interpretation as a gravitational constant:

$$K = e^{-(n+2)\Psi}$$

8. The existence of a cosmological constant Φ depending on Ψ with an asymptotic behavior for large μ

$$\Phi(\mu) \sim e^{2(n+2)\Psi}\frac{\text{const}}{\mu} \qquad (\text{or} \quad \sim\text{const})$$

9. A new term in the equation of motion for a test particle with a scalar force:

$$-\frac{\|q\|^2}{8m_0^2}\tilde{g}^{(\alpha\beta)}\left(\frac{1}{\rho^2}\right)_{,\beta}$$

in terms of a scalar field ρ, or

$$-\frac{\|q\|^2}{4m_0^2}\tilde{g}^{(\alpha\beta)}\,e^{2\Psi}\Psi_{,\beta}$$

in terms of a scalar field Ψ. This force has short range, as do the fields Ψ and ρ.

10. A Lagrangian for the scalar field Ψ and its interaction with the Yang-Mills field in which the Lagrangian of the Yang-Mills field plays the role of a mass term for the scalar field.

11. An energy-momentum tensor for the scalar field Ψ with nonzero trace.

12. Results 10 and 11 suggest that the scalar field Ψ is massive and has Yukawa-type behavior with short range.

All of these effects vanish if the skew-symmetric part of the metric is zero. We get classical results from Jordan-Thiry theory in the non-Abelian case with the propagation of the scalar field Ψ if $n \geq 2$ ($n = \dim G$) and with an enormous cosmological constant which has in front a factor depending on Ψ.

Finally, let us write the full Lagrangian of the theory using CGS units. One writes

$$\mathscr{L} = \bar{R}(\bar{W}) + 2\pi(\lambda\mu)^2\,e^{-(n+2)\Psi}\mathscr{L}_{\mathrm{YM}} + \frac{8\pi G_N}{c^4}\left(\frac{1}{8\pi}\mathscr{L}_{\mathrm{scal}}(\Psi)\right) + 4\,e^{(2+n)\Psi}\frac{\tilde{R}(\tilde{\Gamma})}{\lambda^2} \tag{10.36}$$

where

$$\mu = \frac{g}{\hbar c}, \qquad \lambda = \frac{2l_{pl}}{\alpha_g^{1/2}}, \qquad \alpha_g = \frac{g^2}{\hbar c} \tag{10.37}$$

Note that the Killing-Cartan tensor h is proportional to the Tr tensor and in general the coefficient is not equal to -1. Thus, we really should redefine λ in such a way that $\lambda^2 \rightarrow \lambda^2\nu$, where

$$h = -\nu\,\mathrm{Tr}$$

The Tr tensor is commonly used to define the Yang-Mills Lagrangian.

Let us remark at the end of this section that we have three equivalent forms of the energy-moment tensor for a gauge field in our theory. Let us write them down:

$$\overset{\text{gauge(1)}}{T_{\alpha\beta}} = -\frac{l_{ab}}{4\pi}\left\{g_{\gamma\beta}g^{\tau\rho}g^{\varepsilon\gamma}L^{a}{}_{\rho\alpha}L^{b}{}_{\tau\varepsilon} - 2g^{[\mu\nu]}H^{(a}{}_{\mu\nu}H^{b)}{}_{\alpha\beta} - \frac{1}{4}g_{\alpha\beta}[L^{a\mu\nu}H^{b}{}_{\mu\nu} - 2(g^{[\mu\nu]}H^{a}{}_{\mu\nu})(g^{[\gamma\sigma]}H^{b}{}_{\gamma\sigma})]\right\} \tag{10.38}$$

$$\overset{\text{gauge(2)}}{T_{\mu\nu}} = -\frac{l_{ab}}{4\pi}\left\{g^{\alpha\beta}L^{a}{}_{\beta\nu}L^{b}{}_{\alpha\mu} - 2g^{[\alpha\beta]}H^{(a}{}_{\alpha\beta}H^{b)}{}_{\gamma\sigma} - \frac{1}{4}g_{\mu\nu} \times[L^{a\alpha\beta}L^{b}{}_{\alpha\beta} - 2(g^{[\alpha\beta]}H^{a}{}_{\alpha\beta})(g^{[\gamma\sigma]}H^{b}{}_{\gamma\sigma})]\right\} + \frac{1}{8\pi}J_{\mu\nu} \tag{10.39}$$

where

$$J_{\mu\nu} = 4l_{ab}(L^{a}{}_{\alpha\mu}L^{b}{}_{\beta\nu}g^{[\alpha\beta]} - L^{a}{}_{\alpha\mu}L^{b}{}_{\tau\varepsilon}g^{\tau\alpha}g_{\beta\nu}g^{[\varepsilon\beta]}) \tag{10.40}$$

and

$$\overset{\text{gauge(3)}}{T_{\alpha\beta}} = -\frac{l_{ab}}{4\pi}\left\{g_{\sigma\beta}L^{a\mu\sigma}H^{a}{}_{\mu\sigma} - 2g^{\mu\nu}H^{(a}{}_{\mu\nu}H^{b)}{}_{\alpha\beta} - \frac{1}{4}g_{\alpha\beta}[L^{a\mu\nu}H^{b}{}_{\mu\nu} - 2(g^{[\mu\nu]}H^{a}{}_{\mu\nu})(g^{[\gamma\sigma]}H^{b}{}_{\gamma\sigma})]\right\} \tag{10.41}$$

It is easy to see that

$$g^{\alpha\beta}\overset{\text{gauge(1)}}{T_{\alpha\beta}} = g^{\alpha\beta}\overset{\text{gauge(2)}}{T_{\alpha\beta}} = g^{\alpha\beta}\overset{\text{gauge(3)}}{T_{\alpha\beta}} = 0 \tag{10.42}$$

$T^{\text{gauge(1)}}_{\alpha\beta}$ has been considered in this section an energy-momentum tensor for a gauge field. They are equivalent *modulo* equation (4.10) and are analogous to the three kinds of energy-momentum tensors for an electromagnetic field.

Let us define two Ad-type 2-forms with values in the Lie algebra $\mathfrak{g}$ (of G),

$$\bar{L} = \tfrac{1}{2}L^{a}{}_{\mu\nu}\theta^{\mu} \wedge \theta^{\nu}X_{a}$$
$$\bar{L} = \tfrac{1}{2}M^{a}{}_{\mu\nu}\theta^{\mu} \wedge \theta^{\nu}X_{a}$$

One easily writes

$$\bar{L} = \Omega - 4\pi\bar{M} = \Omega - \tfrac{1}{2}Q \tag{10.43}$$

where

$$Q = \tfrac{1}{2}Q^{a}{}_{\mu\nu}\theta^{\nu} \wedge \theta^{\nu}X_{a} \tag{10.44}$$

Equations (10.43)-(10.44) give a geometrical interpretation of the form $\bar{L}$, i.e., a Yang-Mills induction 2-form in terms of the curvature and torsion in additional dimensions (gauge dimensions). This is similar to ref. 18.

Let us consider equation (10.14a) in a three-dimensional notation. One gets similarly

$$(aJl_{cd}+l_{cd}A)\cdot \mathbf{D}^d + l_{cd}(\mathbf{V}\times\mathbf{H}^d) = 2l_{cd}A\cdot\mathbf{E}^d$$
$$(bJl_{cd}+l_{dc}K)\cdot \mathbf{D}^d - l_{dc}(\mathbf{W}\times\mathbf{H}^d) = 2bl_{cd}\mathbf{E}^d - 2l_{cd}\mathbf{W}\times\mathbf{B}^d \quad (10.45)$$
$$(\mathbf{V}l_{cd}-\mathbf{Q}l_{dc})\mathbf{D}^d = 2l_{cd}\mathbf{V}\cdot\mathbf{E}^d$$

and

$$(l_{cd}K * \bar{H}^d - l_{dc}A * \bar{H}^d) + (l_{cd}\mathbf{W}\otimes\mathbf{D}^d - l_{dc}\mathbf{U}\otimes\mathbf{D}^d)$$
$$= 2l_{cd}K * \bar{F}^d + 2l_{cd}\mathbf{W}\otimes\mathbf{E}^d \quad (10.46)$$

where for **W**, **U**, **V**, **Q**, a, b, K, and A we have the following notations:

$$a = (g^{-1}g^T)_{44}, \qquad b = (g^Tg^{-1})^T_{44}$$
$$\mathbf{V} = ((g^Tg^{-1})_{4\bar{c}}), \qquad \bar{c} = 1,2,3$$
$$\mathbf{W} = ((g^{-1}g^T)_{4\bar{a}}), \qquad \bar{a} = 1,2,3$$
$$\mathbf{U} = ((g^Tg^{-1})_{\bar{b}4}), \qquad \bar{b} = 1,2,3$$
$$\mathbf{Q} = ((g^{-1}g^T)_{\bar{c}4}), \qquad \bar{c} = 1,2,3$$
$$A^T = ((g^Tg^{-1})_{\bar{a}\bar{c}}), \qquad \bar{a},\bar{c} = 1,2,3$$
$$K = ((g^{-1}g^T)_{\bar{c}\bar{b}}), \qquad \bar{c},\bar{b} = 2,3,4$$

and $*$ means matrix multiplication in three-dimensional space, $\otimes$ means the tensor product of three-dimensional vectors, $\cdot$ means scalar product in three-dimensional Euclidean space, and $A\mathbf{E}$ means the action of a 3×33 matrix on a three-dimensional vector. For the remaining symbols we have

$$\mathbf{E}^d = (E^d{}_{\bar{a}}) = (H^d{}_{4\bar{a}}), \qquad \bar{a} = 1,2,3$$
$$\mathbf{D}^d = (D^d{}_{\bar{a}}) = (L^d{}_{4\bar{a}}), \qquad \bar{a} = 1,2,3 \quad (10.47)$$
$$\mathbf{F}^d = (H^d{}_{\bar{m}\bar{n}}) = (\varepsilon_{\bar{m}\bar{n}\bar{s}}B^d{}_{\bar{s}}) = -(\bar{F}^T)^d \quad (10.48)$$

and

$$\mathbf{B}^d = (B^d{}_{\bar{s}}) = (\tfrac{1}{2}\varepsilon_{\bar{m}\bar{n}\bar{s}}B^d{}_{\bar{s}}) \quad (10.48a)$$
$$\mathbf{H}^d = (H^d{}_{\bar{s}}) = (\tfrac{1}{2}\varepsilon_{\bar{m}\bar{n}\bar{s}}L^d{}_{\bar{m}\bar{n}})$$

In this way we lose the covariance of (10.14a), but we get the relations between three-dimensional vectors $(\mathbf{E}^d,\mathbf{B}^d)$ and $(\mathbf{D}^d,\mathbf{H}^d)$ as in an anisotropic Yang–Mills dielectric medium.

Let us suppose that $\mathbf{D}^d = 0$ and $\mathbf{E}^d \neq 0$. One gets

$$\mathbf{V}\cdot\mathbf{E}^d = 0 \quad (10.49)$$

i.e., $\mathbf{E}^d \perp \mathbf{V}$ and

$$\mathbf{V}\cdot\mathbf{H}^d = 2A\cdot\mathbf{E}^d \tag{10.50}$$

$$l_{dc}(\mathbf{W}\times\mathbf{H}^d) = 2l_{cd}(b\mathbf{E}^d - \mathbf{W}\times\mathbf{B}^d) \tag{10.51}$$

$$(l_{cd}K * \bar{H}^d - l_{dc}A * \bar{H}^d) = 2l_{cd}(K * \bar{F}^d + \mathbf{W}\otimes\mathbf{E}^d) \tag{10.52}$$

Equations (10.49)–(10.52) can be considered the consistency equations for a dielectric confinement solution of the field equations, i.e., $\mathbf{D}^d = 0$ and $\mathbf{E}^d \neq 0$ (nonzero electric field and zero color-charge distribution). Moreover, in our theory there is a different tensor L, i.e.,

$$L^{d\mu\alpha} = g^{\beta\mu}g^{\gamma\alpha}L^d{}_{\beta\gamma} \tag{10.53}$$

and this tensor enters the second pair of Yang–Mills equations in our theory [see equation (10.6)]. Thus, we can connect vectors $\mathbf{D}^d$ and $\mathbf{H}^d$ to this tensor,

$$\mathbf{D}^d = (D^d{}_{\bar{a}}) = (L^d{}_{\bar{a}4})$$
$$\mathbf{H}^d = (H^d{}_{\bar{s}}) = (\tfrac{1}{2}\varepsilon^{\bar{s}\bar{m}\bar{n}}L^{d\bar{m}\bar{n}})$$

In this case we should rewrite equation (10.14) in terms of $L^{d\mu\nu}$. One gets

$$l_{dc}g_{\mu\beta}g_{\alpha\rho}L^{d\nu\rho} + l_{cd}g_{\alpha\delta}g^{\delta\gamma}g_{\gamma\rho}g_{\beta\mu}L^{d\mu\rho} = 2l_{cd}g_{\alpha\delta}g^{\delta\gamma}H^d{}_{\beta\gamma} \tag{10.54}$$

One gets

$$\begin{aligned}(2l_{dc}g_{44}g_{[\bar{m}4]} &+ l_{cd}g_{4\delta}g^{\delta\gamma}(g_{\gamma4}g_{4\bar{m}} - g_{44}g_{\gamma\bar{m}}))D^d{}_{\bar{m}}\\ &+ (l_{dc}g_{4\bar{r}}g_{\bar{m}4} - l_{cd}g_{4\bar{r}}g_{4\delta}g^{\delta\gamma}g_{\gamma\bar{m}})H^d{}_{\bar{m}\bar{r}}\\ &= -2l_{cd}g_{4\delta}g^{\delta\bar{c}}E^d{}_{\bar{c}}\end{aligned} \tag{10.55}$$

$$\begin{aligned}(l_{dc}(g_{44}g_{\bar{m}\bar{b}} - g_{4\bar{b}}g_{4\bar{m}}) &+ l_{cd}(g_{4\delta}g^{\delta\gamma}g_{\gamma4}g_{\bar{b}\bar{m}} - g_{4\delta}g^{\delta\gamma}g_{\gamma\bar{m}}g_{\bar{b}4}))D^d{}_{\bar{m}}\\ &+ (l_{dc}g_{\bar{m}\bar{b}}g_{4\bar{r}} + l_{cd}g_{\bar{b}\bar{m}}g_{4\delta}g^{\delta\gamma}g_{\gamma\bar{r}})\bar{H}^d{}_{\bar{m}\bar{r}}\\ &= 2l_{cd}(g_{4\delta}g^{\delta\bar{c}}\bar{F}^d{}_{\bar{b}\bar{c}} - g_{4\delta}g^{\delta4}E^d{}_{\bar{b}})\end{aligned} \tag{10.56}$$

$$\begin{aligned}(l_{dc}(g_{\bar{m}4}g_{\bar{a}4} - g_{44}g_{\bar{a}\bar{m}}) &+ l_{cd}(g_{44}g_{\bar{a}\delta}g^{\delta\gamma}g_{\gamma\bar{m}} + g_{\bar{a}\delta}g^{\delta\gamma}g_{\gamma4}g_{4\bar{m}}))D^d{}_{\bar{m}}\\ &+ (l_{dc}g_{\bar{m}4}g_{\bar{a}\bar{r}} + l_{cd}g_{4\bar{m}}g_{\bar{a}\delta}g^{\delta\gamma}g_{\gamma\bar{r}})\bar{H}^d{}_{\bar{m}\bar{r}}\\ &= -2l_{cd}g_{\bar{a}\delta}g^{\delta\bar{c}}E^d{}_{\bar{c}}\end{aligned} \tag{10.57}$$

$$\begin{aligned}(l_{dc}(g_{\bar{r}\bar{b}}g_{\bar{a}4} - g_{4\bar{b}}g_{\bar{a}\bar{r}}) &+ l_{cd}(g_{\bar{a}\delta}g^{\delta\gamma}g_{\gamma4}g_{\bar{b}\bar{r}} - g_{\bar{a}\delta}g^{\delta\gamma}g_{\gamma\bar{r}}g_{\bar{b}\bar{4}}))D^d{}_{\bar{r}}\\ &+ (l_{dc}g_{\bar{m}\bar{b}}g_{\bar{a}\bar{r}} + l_{cd}g_{\bar{a}\delta}g^{\delta\gamma}g_{\gamma\bar{r}}g_{\bar{b}\bar{m}})\bar{H}^d{}_{\bar{m}\bar{r}}\\ &= 2l_{cd}(g_{\bar{a}\delta}g^{\delta4}E^d{}_{\bar{b}} + g_{\bar{a}\delta}g^{\delta\bar{c}}\bar{F}^d{}_{\bar{b}\bar{c}})\end{aligned} \tag{10.58}$$

Supposing $\mathbf{D}^d = 0$ $(D^d{}_{\bar{r}} = 0)$, one gets

$$(l_{dc}g_{4\bar{r}}g_{\bar{m}4} - l_{cd}g_{4\bar{r}}g_{4\delta}g^{\delta\gamma}g_{\gamma\bar{m}})\bar{H}^d{}_{\bar{m}\bar{n}} = -2l_{cd}g_{4\delta}g^{\delta\bar{c}}E^d{}_{\bar{c}} \tag{10.59}$$

$$(l_{dc}g_{\bar{m}\bar{b}}g_{4\bar{r}} + l_{cd}g_{\bar{b}\bar{m}}g_{\bar{a}\delta}g^{\delta\gamma}g_{\gamma\bar{r}})\bar{H}^d{}_{\bar{m}\bar{r}} = 2l_{cd}(g_{4\delta}g^{\delta\bar{c}}\bar{F}^d{}_{\bar{b}\bar{c}} - g_{4\delta}g^{\delta4}E^d{}_{\bar{b}}) \tag{10.60}$$

$$(l_{dc}g_{\bar{m}4}g_{\bar{a}\bar{r}} + l_{c\bar{d}}g_{4\bar{m}}g_{\bar{a}\delta})\bar{H}^d{}_{\bar{m}\bar{r}} = -2l_{cd}g_{\bar{a}\delta}g^{\delta\bar{c}}E^d{}_{\bar{c}} \tag{10.61}$$

$$(l_{dc}g_{\bar{m}\bar{b}}g_{\bar{a}\bar{r}} + l_{cd}g_{4\bar{m}}g_{\bar{a}\delta}g^{\delta\gamma}g_{\gamma\bar{r}})\bar{H}^d{}_{\bar{m}\bar{r}} = 2l_{cd}(g_{\bar{a}\delta}g^{\delta4}E^d{}_{\bar{b}} + g_{\bar{a}\delta}g^{\delta\bar{c}}\bar{F}^d{}_{\bar{b}\bar{c}}) \tag{10.62}$$

Moreover, we should mention that a separation into space and time components of $g_{\alpha\beta}$, $H^{d}{}_{\mu\nu}$, $L^{d}{}_{\mu\nu}$, and $L^{d\mu\nu}$ is possible only if we deal with the stationary case. We suppose this in order to have a physical interpretation of the condition $D^{d}{}_{\bar{c}}=0$. Otherwise, our considerations have a purely formal character.

Equations (10.59)–(10.62) should be considered the consistency conditions for $D^{d}{}_{\bar{c}}=0$. Thus, we can treat them as equations not only for $\bar{H}^{d}{}_{\bar{m}\bar{n}}$, but also for g_{44}, $g_{4\bar{m}}$, $g_{\bar{m}\bar{n}}$ under stationary conditions (the same conditions for $\bar{H}^{d}{}_{\bar{m}\bar{n}}$, $E^{d}{}_{\bar{c}}$, $\bar{F}^{d}{}_{\bar{b}\bar{c}}$). Thus, the dielectric confinement solution of the field equations can be derived from the second possibility, i.e., for $\mathbf{D}^{d}$ obtained from $L^{d\mu\nu}$. A stationary space-time determines a three-dimensional manifold Σ_3 defined by the smooth map $\Phi: E\to\Sigma_3$, where $\Phi(x)$ denotes the trajectory of the timelike Killing vector $\bar{\eta}$. The elements of Σ_3 are orbits of the one-dimensional group of motions generated by $\bar{\eta}$. The 3-space Σ_3 is called the quotient space E/G_1. There is a one-to-one correspondence between tensor fields on Σ_3 and tensors on E, T satisfying $\bar{\eta}^{\mu}T_{\mu}{}^{\nu}=\bar{\eta}_{\mu}T_{\nu}{}^{\mu}=\mathfrak{L}_{\bar{\eta}}T_{\mu}{}^{\nu}$, where $\bar{\eta}_{\mu}=g_{(\mu\nu)}\bar{\eta}^{\nu}$. In our case we have on Σ_3 the following tensors: $h_{(\mu\nu)}=g_{(\mu\nu)}+(-g_{\alpha\beta}\bar{\eta}^{\alpha}\bar{\eta}^{\beta})^{1/2}\bar{\eta}_{\mu}\bar{\eta}_{\nu}$ and an appropriate tensor built from $g_{[\mu\nu]}$ ($\bar{\eta}=\partial/\partial x^{4}$). The action of the group G_1 can be lifted to the bundle $\underline{P}$ and we get $\mathfrak{L}_{\eta}\omega=\mathfrak{L}_{\eta}\gamma=\mathfrak{L}_{\eta}\Omega=\mathfrak{L}_{\eta}\omega^{A}{}_{B}$, where $\eta=\pi^{*}\eta$. In our case we need a stationary condition for $L^{a}{}_{\mu\nu}$, $H^{a}{}_{\mu\nu}$, $L^{a\mu\nu}$. This can be defined on the bundle $\underline{P}$, $\mathfrak{L}_{\eta}L^{a}{}_{\mu\nu}=\mathfrak{L}_{\eta}H^{a}{}_{\mu\nu}=\mathfrak{L}^{a\mu\nu}=0$, because $L^{a}{}_{\mu\nu}$, $H^{a}{}_{\mu\nu}$ are defined on $\underline{P}$, *not* on E. In order to consider these conditions on E, we should take a local section e of $\underline{P}$ and define a homomorphism $\sigma: G_1\to G$ such that

$$\hat{r}^{*}(h)F^{a}{}_{\mu\nu}=U^{a}{}_{b}(\sigma(h^{-1}))F^{b}{}_{\mu\nu} \tag{10.63}$$

and the same for $B^{a}{}_{\mu\nu}=e^{*}L^{a}{}_{\mu\nu}$, $A^{a\mu\nu}=e^{*}L^{a\mu\nu}$, where $F^{a}{}_{\mu\nu}=e^{*}H^{a}{}_{\mu\nu}$, and $\hat{r}^{*}$ means the action of the group G_1 lifted to the bundle $\underline{P}$. It seems, however, that the best choice is to consider tensors on a space Σ_{3+n} similarly as tensors on Σ_3. In this way equations (10.55)–(10.62) have a correct meaning on Σ_{3+n}. The Σ_{3+n} is a smooth manifold of orbits of the one-dimensioned group G_1 generated by $\pi^{*}\eta=\eta$. In the case of the static field configuration there is a natural way of introducing subspaces P_{3+n} (orthogonal to the Killing trajectories).

11. SPECIAL CASES

Let us consider some special cases of the theory. First of all let $g_{\alpha\beta}$ be symmetric and $l_{ab}\neq l_{ba}$. In this case we are able to solve equation (10.14) and we get

$$L^{a}{}_{\beta\gamma}=h^{ac}l_{cd}H^{d}{}_{\beta\gamma} \tag{11.1}$$

The Yang–Mills Lagrangian takes the form

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{8\pi}[h_{bd} + \mu^2 k^c{}_b k_{cd})H^{b\mu\nu}H^d{}_{\mu\nu} \tag{11.2}$$

where $k^c{}_d = h^{ce}k_{ed}$. Let us suppose that $l_{ab} = h_{ab}$ and $g_{\alpha\beta} \neq g_{\beta\alpha}$. In this case we have

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{8\pi} h_{ab}(2H^a H^b - L^{a\mu\nu}H^b{}_{\mu\nu}) \tag{11.3}$$

where $H^a = g^{[\mu\nu]}H^a{}_{\mu\nu}$ and the relationship between $L^a{}_{\alpha\beta}$ and $H^a{}_{\alpha\beta}$ is

$$g_{\mu\beta}g^{\gamma\mu}L^a{}_{\gamma\alpha} + g_{\alpha\mu}g^{\mu\gamma}L^a{}_{\beta\gamma} = 2g_{\alpha\mu}g^{\mu\gamma}H^a{}_{\beta\gamma} \tag{11.4}$$

Now there is no mixing in the gauge indices (no mixing of "color charges"). In the first special case we are able to calculate the polarization tensor $M^a{}_{\alpha\mu}$ and we get

$$M^a{}_{\alpha\beta} = \frac{1}{4\pi}(\delta^a{}_d - h^{ac}l_{cd})H^d{}_{\alpha\beta} \tag{11.5}$$

In the first case we are able to make the cosmological constant as small as we need. In the second case we get the classical result with enormous cosmological constant. Let us notice that this case corresponds to the situation with $\mu = 0$ or to the case $l_{ab} = h_{ab}$ mentioned in Section 3. If $G = U(1)$, we get

$$\mathscr{L}_{\mathrm{em}} = \frac{1}{8\pi}[2(g^{[\mu\nu]}F_{\mu\nu})^2 - H^{\mu\nu}F_{\mu\nu}] \tag{11.6}$$

where

$$g_{\mu\beta}g^{\gamma\mu}H_{\gamma\alpha} + g_{\alpha\mu}g^{\mu\gamma}H_{\beta\gamma} = 2g_{\alpha\mu}g^{\mu\gamma}F_{\beta\gamma} \tag{11.7}$$

and we do not obtain the cosmological constant [$U(1)$ is Abelian].

Let us come back to the Lagrangian (11.2) and let us substitute

$$k_{ab} = C^c{}_{ab}V_c \tag{11.8}$$

from Section 2. One gets

$$\mathscr{L}_{\mathrm{YM}} = \frac{1}{8\pi}\left(1 + \mu^2 \frac{\|V\|^2}{n-1}\right) h_{bd}H^{b\mu\nu}H^d{}_{\mu\nu} + \frac{1}{8\pi}\frac{\mu^2}{n-1} V_b H^{b\mu\nu} V_d H^d{}_{\mu\nu} \tag{11.9}$$

where

$$\|V\|^2 = -h_{ab}V^a V^b = \mathrm{const} > 0$$

is a square of the norm of the vector V. Thus, we get the sum of the usual Yang–Mills Lagrangian (except the factor in front) and a new term

$$\frac{\mu^2}{n-1} V_b H^{b\mu\nu} V_d H^d{}_{\mu\nu} \tag{11.10}$$

Let us now consider the first term in equation (11.9) for $G = SO(3)$. Thus, one gets

$$-\frac{2\lambda^2}{8\pi}(1+\mu^2)\,\mathrm{Tr}(H^{\mu\nu}H_{\mu\nu}) \tag{11.11}$$

Moreover, in any gauge e (a local section) one finds

$$e^*\omega = \frac{g}{\hbar c} A^a{}_\mu \bar{\theta}^\mu X_a \tag{11.12}$$

where g is a coupling constant. For the generators X_a in an adjoint representation (Ad'_G) we use the following normalization condition: $\mathrm{Tr}(\{X_a, X_b\}) = 2\delta_{ab}$, where $\{\cdot\,,\cdot\}$ means an anticommutator of matrices. Thus, for the strength of the Yang–Mills field one derives

$$e^*H_{\mu\nu} = \frac{g}{\hbar c} F_{\mu\nu} \tag{11.13}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{g}{\hbar c}[A_\mu, A_\nu] \tag{11.14}$$

Formula (11.11) can be rewritten

$$-\frac{1}{8\pi}\frac{2\lambda^2 g^2}{\hbar^2 c^2}(1+\mu^2)\,\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \tag{11.15}$$

Moreover, we should get a factor $8\pi G_N/c^4$. Thus, one finally finds

$$\lambda = 2l_{\mathrm{Pl}} \frac{1}{[\alpha_g(1+\mu^2)]^{1/2}} \tag{11.16}$$

where $l_{\mathrm{Pl}} = (\hbar G_N/c^3)^{1/2}$ is Planck's length and $\alpha_g = g^2/\hbar c$ is a dimensionless coupling constant for the Yang–Mills field.

12. LINEARIZATION OF THE NONSYMMETRIC NON-ABELIAN KALUZA–KLEIN THEORY

Let us consider (10.14) and rewrite it in a more convenient form,

$$\begin{aligned} &l_{dc}(\delta^\gamma{}_\beta - 2g_{[\beta\delta]}g^{\gamma\delta})L^d{}_{\gamma\alpha} + l_{cd}(\delta^\gamma{}_\alpha - 2g_{[\delta\alpha]}g^{\delta\gamma})L^d{}_{\beta\gamma} \\ &\quad = 2(\delta^\gamma{}_\alpha - 2g_{[\delta\alpha]}g^{\delta\gamma})H^d{}_{\beta\gamma}l_{cd} \end{aligned} \tag{12.1}$$

In this section we will use a different notation for a Killing-Cartan tensor on the group G, i.e., P_{ab} in the place of h_{ab}.

Let us expand $L^{d}{}_{\beta\gamma}$ into a power series with respect to $h_{\alpha\beta} = g_{\alpha\beta} - \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is a Minkowski tensor

$$L^{d}{}_{\beta\gamma} = \overset{(0)}{L}{}^{d}{}_{\beta\gamma} + \overset{(1)}{L}{}^{d}{}_{\beta\gamma} + \overset{(2)}{L}{}^{d}{}_{\beta\gamma} + \cdots \tag{12.2}$$

Using the formulas

$$g^{\mu\alpha}g_{\nu\alpha} = (\eta^{\mu\alpha} + \overset{1}{h}{}^{\mu\alpha} + \overset{2}{h}{}^{\mu\alpha} + \cdots)(\eta_{\nu\alpha} + h_{\nu\alpha}) = \delta^{\mu}_{\nu}$$

$$\overset{1}{h}{}^{\mu\nu} = -\eta^{\mu\alpha}\eta^{\nu\beta}h_{\beta\alpha}$$

$$\overset{2}{h}{}^{\mu\nu} = -\eta^{\nu\beta}\overset{1}{h}{}^{\mu\alpha}h_{\beta\alpha} = \eta^{\nu\beta}\eta^{\mu\gamma}\eta^{\alpha\sigma}h_{\alpha\gamma}h_{\beta\alpha}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\beta\alpha} + \eta^{\mu\gamma}\eta^{\nu\sigma}\eta^{\alpha\beta}h_{\beta\gamma}h_{\sigma\alpha} + \cdots$$

one gets up to the second order

$$\begin{aligned} &l_{dc}L^{d}{}_{\beta\alpha} + l_{cd}L^{d}{}_{\beta\alpha} - 2(\eta^{\gamma\delta} - \eta^{\delta\sigma}\eta^{\delta\tau}h_{\tau\sigma} + \eta^{\gamma\rho}\eta^{\delta\varepsilon}\eta^{\sigma\tau}h_{\sigma\rho}h_{\varepsilon\tau}) \\ &\quad\times(h_{[\beta\delta]}l_{dc}L^{d}{}_{\gamma\alpha} + h_{[\gamma\alpha]}l_{cd}L^{d}{}_{\beta\delta}) \\ &= 2l_{cd}H^{d}{}_{\beta\alpha} - 4h_{[\delta\alpha]}l_{cd}H^{d}{}_{\beta\gamma} \\ &\quad\times(\eta^{\delta\gamma} - \eta^{\delta\tau}\eta^{\gamma\sigma}h_{\sigma\tau} + \eta^{\delta\varepsilon}\eta^{\gamma\rho}\eta^{\sigma\tau}h_{\sigma\varepsilon}h_{\rho\tau}) \end{aligned} \tag{12.3}$$

One easily finds

$$\overset{(0)}{L}{}^{\alpha}{}_{\beta\alpha} = p^{ab}l_{bc}H^{c}{}_{\beta\alpha} = H^{a}{}_{\beta\alpha} + \mu p^{ab}k_{bc}H^{c}{}_{\beta\alpha} \tag{12.4}$$

$$\begin{aligned} \overset{(1)}{L}{}^{a}{}_{\beta\alpha} &= \eta^{\delta\gamma}p^{ac}p^{bd}l_{be}(h_{[\delta\alpha]}l_{cd}H^{e}{}_{\beta\gamma} + h_{[\beta\delta]}l_{dc}H^{e}{}_{\gamma\alpha}) \\ &\quad - 2h_{[\delta\alpha]}p^{ac}l_{cd}H^{d}{}_{\beta\gamma}\eta^{\delta\gamma} \end{aligned} \tag{12.5}$$

or

$$\overset{(1)}{L}{}^{a}{}_{\beta\alpha} = (\delta^{a}{}_{e} - \mu^{2}p^{ab}k_{bc}p^{cd}k_{de})\eta^{\gamma\delta}(h_{[\beta\delta]}H^{e}{}_{\gamma\alpha} - h_{[\alpha\delta]}H^{e}{}_{\gamma\beta}) \tag{12.6}$$

$$\begin{aligned} \overset{(2)}{L}{}^{a}{}_{\beta\alpha} &= \eta^{\gamma\delta}\eta^{\tau\sigma}h_{(\gamma\tau)}(k^{a}_{+b}k^{b}_{-c})(h_{[\alpha\beta]}H^{c}{}_{\sigma\beta} - h_{[\beta\delta]}H^{c}{}_{\sigma\alpha}) \\ &\quad + \mu\eta^{\gamma\delta}\eta^{\tau\sigma}(k^{a}_{+b}k^{b}_{-c})p^{cr}k_{rd} \\ &\quad \times(h_{[\gamma\tau]}h_{[\alpha\delta]}H^{d}{}_{\alpha\beta} - h_{[\gamma\tau]}h_{[\beta\delta]}H^{d}{}_{\sigma\alpha} + 2h_{[\beta\tau]}h_{[\delta\alpha]}H^{d}{}_{\sigma\gamma}) \end{aligned} \tag{12.7}$$

where

$$p^{ar}p_{br} = \delta^{a}{}_{b} \tag{12.8}$$

$$k^{a}_{\pm b} = \delta^{a}{}_{b} \pm \mu p^{ar}k_{rb} \tag{12.8a}$$

Using equation (11.8), one easily derives that

$$k^a_{+b}k^b_{-c}=\left(1+\frac{\mu^2\|V\|^2}{n-1}\right)\delta^a_c+\frac{\mu^2}{n-1}V^aV_c$$

Let us consider the Lagrangian for the Yang–Mills field in the nonsymmetric non-Abelian Kaluza–Klein theory. We have [see equation (10.21)]

$$\begin{aligned}\mathscr{L}_{\rm YM}=\frac{1}{8\pi}[&-2p_{ab}(g^{[\alpha\beta]}H^a{}_{\alpha\beta})(g^{[\mu\nu]}H^b{}_{\mu\nu})\\&+(p_{ab}+\mu k_{ab})g^{\alpha\mu}g^{\beta\nu}L^a{}_{\alpha\beta}H^b{}_{\mu\nu}]\end{aligned}\tag{12.9}$$

Let us expand (12.9) into a power series with respect to $h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$,

$$\mathscr{L}_{\rm YM}=\overset{(0)}{\mathscr{L}}_{\rm YM}+\overset{(1)}{\mathscr{L}}_{\rm YM}+\overset{(2)}{\mathscr{L}}_{\rm YM}+\cdots\tag{12.10}$$

Using (12.9) and (12.4)–(12.7), one gets after some calculations

$$8\pi\overset{(0)}{\mathscr{L}}_{\rm YM}=(p_{cb}+\mu^2k_{cr}p^{rs}k_{sb})H^c{}_{\beta\gamma}H^b{}_{\mu\alpha}\eta^{\beta\mu}\eta^{\gamma\alpha}\tag{12.11}$$

$$\begin{aligned}8\pi\overset{(1)}{\mathscr{L}}_{\rm YM}=&-2(p_{ab}+\mu k_{ab})\eta^{\beta\sigma}\eta^{\mu\tau}\eta^{\gamma\alpha}[h_{(\tau\sigma)}H^a{}_{\beta\gamma}H^b{}_{\mu\alpha}\\&+(\mu p^{ar}k_{rs})(h_{(\tau\sigma)}+h_{[\tau\sigma]})H^s{}_{\beta\gamma}H^b{}_{\mu\alpha}\\&+\mu^2p^{ar}k_{rp}p^{pq}k_{qs}h_{[\tau\sigma]}H^s{}_{\beta\gamma}H^b{}_{\mu\alpha}]\end{aligned}\tag{12.12}$$

$$\begin{aligned}8\pi\overset{(2)}{\mathscr{L}}_{\rm YM}=&(p_{ab}+\mu k_{ab})H^s{}_{\beta\gamma}H^b{}_{\mu\alpha}h_{\tau\sigma}h_{\delta\varepsilon}\\&\times\{[2(\eta^{\beta\varepsilon}\eta^{\mu\tau}\eta^{\sigma\delta}\eta^{\alpha\gamma})+\eta^{\beta\sigma}\eta^{\mu\tau}\eta^{\gamma\varepsilon}\eta^{\alpha\delta}]k^a_{+s}\\&+(\eta^{\varepsilon\sigma}\eta^{\mu\tau}\eta^{\gamma\alpha}\eta^{\beta\delta}-\eta^{\delta\sigma}\eta^{\mu\tau}\eta^{\gamma\alpha}\eta^{\beta\varepsilon}\\&+\eta^{\gamma\sigma}\eta^{\mu\tau}\eta^{\varepsilon\alpha}\eta^{\beta\delta}-\eta^{\gamma\sigma}\eta^{\mu\tau}\eta^{\delta\alpha}\eta^{\beta\varepsilon})\\&\times(k^a_{+r}k^r_{-s})+\tfrac12(\eta^{\gamma\mu}\eta^{\tau\alpha}\eta^{\delta\beta}\eta^{\varepsilon\sigma}+\eta^{\gamma\mu}\eta^{\tau\alpha}\eta^{\varepsilon\beta}\eta^{\delta\sigma}\\&-\eta^{\gamma\mu}\eta^{\sigma\alpha}\eta^{\delta\beta}\eta^{\varepsilon\tau}-\eta^{\gamma\mu}\eta^{\sigma\alpha}\eta^{\varepsilon\beta}\eta^{\delta\tau})(k^a_{+r}k^r_{-s})\\&+\tfrac12\mu(-\eta^{\gamma\mu}\eta^{\tau\alpha}\eta^{\delta\beta}\eta^{\varepsilon\sigma}+\eta^{\gamma\mu}\eta^{\tau\alpha}\eta^{\varepsilon\beta}\eta^{\delta\sigma}\\&+\eta^{\gamma\mu}\eta^{\sigma\alpha}\eta^{\delta\beta}\eta^{\varepsilon\tau}-\eta^{\gamma\mu}\eta^{\sigma\alpha}\eta^{\varepsilon\beta}\eta^{\delta\tau})(k^a_{+r}k^r_{-p})p^{pq}k_{qs}\\&+\tfrac12\mu(\eta^{\gamma\delta}\eta^{\tau\beta}\eta^{\sigma\mu}\eta^{\varepsilon\alpha}-\eta^{\gamma\delta}\eta^{\sigma\beta}\eta^{\tau\mu}\eta^{\varepsilon\alpha}\\&-\eta^{\gamma\varepsilon}\eta^{\tau\beta}\eta^{\sigma\mu}\eta^{\delta\alpha}+\eta^{\gamma\varepsilon}\eta^{\sigma\beta}\eta^{\tau\mu}\eta^{\delta\alpha})(k^a_{+r}k^r_{-p})p^{pq}k_{qs}\\&-\tfrac12(\eta^{\mu\sigma}\eta^{\alpha\tau}\eta^{\beta\varepsilon}\eta^{\gamma\delta}-\eta^{\mu\tau}\eta^{\alpha\sigma}\eta^{\beta\varepsilon}\eta^{\gamma\delta}\\&-\eta^{\mu\sigma}\eta^{\alpha\tau}\eta^{\beta\delta}\eta^{\gamma\varepsilon}+\eta^{\mu\sigma}\eta^{\alpha\tau}\eta^{\beta\delta}\eta^{\gamma\varepsilon})\delta^a{}_s\}\end{aligned}\tag{12.13}$$

In the first order of approximation in $h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$ one gets

$$\begin{aligned}
\mathscr{L}_{\mathrm{YM}}=&\frac{1}{8\pi}(p_{cb}+\mu^2 k_{cr}h^{rs}k_{sb})H^c{}_{\beta\gamma}H^b{}_{\mu\alpha}\eta^{\beta\mu}\eta^{\gamma\alpha}\\
&-\frac{1}{4\pi}(p_{ab}+\mu^2 k_{ar}h^{rs}k_{sb})\eta^{\beta\sigma}\eta^{\mu\tau}\eta^{\gamma\alpha}h_{(\tau\sigma)}H^a{}_{\beta\gamma}H^b{}_{\mu\alpha}\\
&+\frac{\mu}{4\pi}h^{ar}k_{rs}(p_{ab}+\mu k_{ab})(\delta^s{}_p+\mu p^{sq}k_{qp})\\
&\times\eta^{\beta\sigma}\eta^{\mu\tau}\eta^{\gamma\alpha}h_{[\tau\sigma]}H^p{}_{\beta\gamma}H^b{}_{\mu\alpha}.
\end{aligned}\tag{12.14}$$

It is easy to see that the last term in equation (12.14) gives a skewon–gluon coupling in the first order of approximation. If $\mu=0$, this term vanishes and there is no skewon–gluon coupling up to the first order of approximation. If $\mu\neq 0$ and $k_{ab}\neq 0$ we get skewon–gluon interactions in the first order of approximation.

We can find interactions between gluons and gravitons. This is similar to the usual case except for a factor in front [see the first term in (12.15)]

$$\begin{aligned}
\overset{(1)}{\mathscr{L}}_{\mathrm{YM}}=-\frac{1}{4\pi}\Bigg[&(p_{ab}+\mu k_{ab})(\delta^a{}_s+\mu p^{ar}k_{rs})\eta^{\beta\sigma}\eta^{\mu\tau}\eta^{\gamma\alpha}h_{(\tau\sigma)}H^s{}_{\beta\gamma}H^b{}_{\mu\alpha}\\
&-\frac{\mu}{4\pi}(p_{ab}+\mu k_{ab})p^{ar}k_{rs}(\delta^s{}_p+\mu p^{sq}k_{qp})\\
&\times\eta^{\beta\sigma}\eta^{\mu\tau}\eta^{\gamma\alpha}h_{[\tau\sigma]}H^p{}_{\beta\gamma}H^b{}_{\mu\alpha}\Bigg]
\end{aligned}\tag{12.15}$$

The skewon–gluon terms are proportional to μ. They are given by the case of coupling an antisymmetric tensor to the Yang–Mills field. The gluon propagator can be given by the zeroth-order term in the Lagrangian,

$$\overset{(0)}{\mathscr{L}}_{\mathrm{YM}}=\frac{1}{8\pi}(p_{cb}+\mu^2 k_{cr}p^{rs}k_{sb})\eta^{\beta\mu}\eta^{\gamma\alpha}H^c{}_{\beta\gamma}H^b{}_{\mu\alpha}\tag{12.16}$$

This differs from the usual case by a factor in front. In the nonsymmetric non-Abelian Kaluza–Klein theory the constant μ is connected to the cosmological constant. Thus, we get a connection between the skewon–gluon interaction and the cosmological constant. In the case of $G=SO(3)$ we get in Section 7 that for $\mu=\mu_0\cong -5.667\cdots$ the cosmological constant vanishes. Thus, we get a coupling constant of the skewon–gluon interaction for $G=SO(3)\simeq SU(2)$. We can choose $|\mu|>10^{127}$. In this case the skewon–gluon interaction is very strong. We also have a solution for other compact, simple Lie groups, dim $G>4$ (see Section 7).

Let us pass to the Lagrangian for the scalar field in our theory,

$$\mathscr{L}_{\text{scal}}(\Psi) = (m\tilde{g}^{(\gamma\nu)} + n^2 g^{[\mu\nu]} g_{\delta\mu} \tilde{g}^{(\delta\gamma)})\Psi_{,\nu}\Psi_{,\gamma} \tag{12.17}$$

This field is uncharged. Let us expand $\mathscr{L}_{\text{scal}}(\Psi)$ into a power series with respect to $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. One gets

$$\mathscr{L}_{\text{scal}} = \overset{(0)}{\mathscr{L}}_{\text{scal}} + \overset{(1)}{\mathscr{L}}_{\text{scal}} + \overset{(2)}{\mathscr{L}}_{\text{scal}} + \cdots \tag{12.18}$$

where

$$\overset{(0)}{\mathscr{L}}_{\text{scal}}(\Psi) = m\eta^{\mu\rho}\Psi_{,\mu}\Psi_{,\rho} \tag{12.19}$$

$$\overset{(1)}{\mathscr{L}}_{\text{scal}}(\Psi) = -(m\eta^{\mu\alpha}\eta^{\rho\beta}h_{(\beta\alpha)})\Psi_{,\mu}\Psi_{,\rho} \tag{12.20}$$

$$\begin{aligned}\overset{(2)}{\mathscr{L}}_{\text{scal}}(\Psi) = [&m\eta^{\mu\gamma}\eta^{\rho\delta}\eta^{\alpha\beta}h_{(\beta\gamma)}h_{(\alpha\delta)} \\ &+ n^2(\eta^{\alpha[\sigma}\eta^{\mu]\beta}\eta^{\rho\gamma}h_{\beta\alpha}h_{(\gamma\sigma)} \\ &+ \eta^{\alpha[\mu}\eta^{\nu]\beta}\eta^{\rho\delta}h_{\beta\alpha}h_{\delta\nu})]\Psi_{,\mu}\Psi_{,\rho}\end{aligned} \tag{12.21}$$

Now we have a nonvanishing $\mathscr{L}^{(0)}_{\text{scal}}(\Psi)$ and the field Ψ propagates even if the skew-symmetric part of $g_{\alpha\beta}$ vanishes, i.e., $h_{[\alpha\beta]} = 0$. This is different than in the electromagnetic case. Let us suppose that the field Ψ is weak:

$$|\Psi| \ll 1 \tag{12.22}$$

One easily gets

$$e^{-(n+2)\Psi} = 1 - (n+2)\Psi + \frac{(n+2)}{2}\Psi^2 + \cdots \tag{12.33}$$

and

$$e^{(n+2)\Psi} = 1 + (n+2)\Psi + \frac{(n+2)^2}{2}\Psi^2 + \cdots \tag{12.24}$$

The field Ψ is the scalar field connected to the gravitational constant. Thus, Ψ is the scalar part of the gravitational field. Our approximation presented here is the approximation up to the second order with respect to $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ and Ψ. In this way one easily gets for the Lagrangian in the nonsymmetric non-Abelian Jordan–Thiry theory (except for the Lagrangian of the pure gravitational field from NGT)

$$\begin{aligned}8\pi\mathscr{L} = 8\pi(&\overset{(0)}{\mathscr{L}}_{\text{YM}} + \overset{(1)}{\mathscr{L}}_{\text{YM}} + \overset{(2)}{\mathscr{L}}_{\text{YM}}) \\ &+ (\mathscr{L}_{\text{scal}} = \overset{(0)}{\mathscr{L}}_{\text{scal}} + \overset{(1)}{\mathscr{L}}_{\text{scal}} + \overset{(2)}{\mathscr{L}}_{\text{scal}}) \\ &- (n+2)\Psi[8\pi(\overset{(0)}{\mathscr{L}}_{\text{YM}} + \overset{(1)}{\mathscr{L}}_{\text{YM}}) - \tilde{R}(\tilde{\Gamma})] \\ &+ \frac{(n+2)^2}{2}\Psi^2(8\pi\overset{(0)}{\mathscr{L}}_{\text{YM}} + \tilde{R}(\tilde{\Gamma}))\end{aligned} \tag{12.25}$$

It is easy to see that in this approximation we get the masslike term for the field Ψ

$$+\frac{(n+2)^2}{2}\Psi^2(8\pi\overset{(0)}{\mathscr{L}}_{\mathrm{YM}}+\tilde{R}(\tilde{\Gamma})) \tag{12.26}$$

and an interaction term

$$-(n+2)\Psi[8\pi(\overset{(0)}{\mathscr{L}}_{\mathrm{YM}}+\overset{(1)}{\mathscr{L}}_{\mathrm{YM}})-\tilde{R}(\tilde{\Gamma})] \tag{12.27}$$

The last expression (12.27) can be treated as the interaction of the field Ψ with source, i.e.,

$$\Psi J \tag{12.28}$$

where

$$J=-(n+2)[8\pi(\overset{(0)}{\mathscr{L}}_{\mathrm{YM}}+\overset{(1)}{\mathscr{L}}_{\mathrm{YM}})-\tilde{R}(\tilde{\Gamma})] \tag{12.29}$$

is an external source for the field Ψ. In the first order of approximation in $h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$ and Ψ one gets

$$\begin{aligned}8\pi\mathscr{L}=&(h_{cb}+\mu^2k_{cr}h^{rs}k_{sb})H^c{}_{\beta\gamma}H^b{}_{\mu\alpha}\eta^{\beta\mu}\eta^{\gamma\alpha}\\&-2(h_{ab}+\mu^2k_{ar}h^{rs}k_{sb})\eta^{\beta\sigma}\eta^{\mu\tau}\eta^{\gamma\alpha}h_{(\tau\sigma)}H^a{}_{\beta\gamma}H^b{}_{\mu\alpha}\\&+2\mu h^{ar}k_{rs}(h_{ab}+\mu k_{ab})(\delta^s{}_p+\mu h^{sq}k_{qp})\\&\times\eta^{\beta\sigma}\eta^{\mu\tau}\eta^{\gamma\alpha}h_{[\tau\sigma]}H^p{}_{\beta\gamma}H^b{}_{\mu\alpha}\\&+m(\eta^{\mu\rho}-\eta^{\mu\alpha}\eta^{\rho\beta}h_{(\beta\alpha)})-\Psi_{,\mu}\Psi_{,\rho}\\&-(n+2)\Psi[(h_{cb}+\mu^2k_{cr}h^{rs}k_{sb})H^c{}_{\beta\gamma}H^b{}_{\mu\alpha}\eta^{\beta\mu}\eta^{\gamma\alpha}-\tilde{R}(\tilde{\Gamma})]\end{aligned} \tag{12.30}$$

i.e., we get an interaction term for the field Ψ and $h_{\beta\alpha}$ enters into the kinetic term for Ψ. We have a skewon interaction with the field Ψ and with the Yang–Mills field in the second order of expansion. The field Ψ interacts with the Yang–Mills field due to the pseudo-mass term. Despite this the field Ψ is uncharged. The propagator of the field Ψ is as usual apart from the factor

$$m=l^{[dc]}l_{[dc]}-3n(n-1) \tag{12.31}$$

This field does not propagate to first order if

$$m=0 \tag{12.32}$$

Let us remark on the convergence of the series appearing here. They are power series with respect to $h_{\mu\nu}$ and they converge for sufficiently small $h_{\mu\nu}$. However, all of the functions of $h_{\mu\nu}$ considered here (i.e., $g^{\mu\nu}$, $L^a{}_{\mu\nu}$, $\mathscr{L}_{\mathrm{YM}}$) are well defined for any $h_{\mu\nu}$. They are rational functions of this variable. Moreover, the exact form of these functions is hard to get.

In this section we found the linear version of the nonsymmetric Jordan–Thiry theory (see refs. 28 and 29). We found the Lagrangian up to the second order of approximation with respect to the gravitational field in this theory. We recall that in the electromagnetic case we found that the scalar field Ψ does not propagate in the first order of approximation. Due to this we find that there is no scalar (monopole) radiation to this order. Simultaneously, one concludes that in the first order of approximation the theory has nonvarying effective gravitational constant. This means that the variation of the gravitational constant is at least an effect of the second order in this theory.

In the general non-Abelian case we find that in the first order of approximation the scalar field propagates and couples only to the symmetric part of the metric. Due to this we find that the scalar field propagates in the first order of approximation. However, this field couples to the cosmological constant and to the Lagrangian for the Yang–Mills field. Thus, it seems that this field is massive (pseudo-mass terms). Simultaneously, the trace of an energy-momentum tensor for this field is not zero. This indicates that this field is massive and has Yukawa-type behavior. Thus, it seems that there is no long-range radiation connected with the scalarons, which are massive. In order to prove this, it is necessary to find an exact solution of the field equations in the spherical, static case (similarly as for the Kaluza–Klein theory; see ref. 30) with Yukawa-type behavior for the field Ψ. Unfortunately, such a solution is unknown.

It is easy to answer what the spin content of the theory is (in the linear approximation). In the electromagnetic case it is (2, 0, 1)—a graviton, a skewon, a photon, *no* scalaron (see ref. 29). In the non-Abelian case

$$(2, 0, \underbrace{1, 1, \ldots, 1}_{n \text{ times}}, 0)$$

i.e., a graviton, a skewon, n gauge bosons (n gluons, intermediate bosons), and a scalaron. It is interesting to consider in more detail the interaction between the skewon field and the gauge boson (gluon, intermediate boson) field. This is an interaction between the generalized Maxwell field $h_{[\mu\nu]}$ and the non-Abelian gauge field. In the linear approximation of NGT, $h_{[\mu\nu]}$ enters the theory via its strength $\bar{F}_{\mu\nu\lambda} = \partial_{[\lambda} h_{[\mu\nu]]}$ carrying a zero spin. For the Abelian gauge field, $h_{[\mu\nu]}$ is connected to the Kalb–Ramond string field (see ref. 66). It would be interesting to examine this interaction as a possibility for confinement for gluons in the case $G = SU(3)$. It is also interesting to examine this term in the context of string theory, or even superstrings.

Let us consider the Lagrangian for the scalar field in the electromagnetic case (five-dimensional) (see ref. 29). For the spherically symmetric solution in the nonsymmetric Kaluza–Klein theory of gravitation we have that

$$g^{[14]}=\frac{l^2}{r^2} \tag{12.33}$$

where l^2 is a constant proportional to fermion number in Moffat's theory of gravitation. The other components of $g^{[\mu\nu]}$ are zero; thus, the constant $(l/l_{\rm Pl})^2$ in this particular case plays the role of a coupling constant between scalarons and the metric, $l_{\rm Pl}$ being the Planck length

$$l_{\rm Pl}=\left(\frac{G_N\hbar}{c^3}\right)^{1/2}$$

This is similar to Brans–Dicke theory, where we have a constant ω (see refs. 93 and 94). Now $(l/l_{\rm Pl})^2$ plays a similar role to w in the nonsymmetric Jordan–Thiry theory. If $l^2=0$, the scalar field Ψ really disappears, similarly as for $\omega=0$ in the Brans–Dicke theory. However, the experimental predictions of NGT theory are very different from those in Brans–Dicke theory (see refs. 65 and 79).

The last question which we can pose here is the problem of ghosts and tachyons in the nonsymmetric Jordan–Thiry theory. We know that the real version of the nonsymmetric theory of gravitation avoids ghosts and tachyons (see refs. 108 and 109). In the linear version of the nonsymmetric Jordan–Thiry theory (the Lagrangian is quadratic with respect to all fields) we have no ghosts and tachyons in the particle spectrum of gravitons, skewons, or gauge bosons. The only problem is the scalar field Ψ. In the electromagnetic case this field disappears in the zeroth order of approximation. Thus, the five-dimensional electromagnetic case avoids ghosts and tachyons. In the $(n+4)$-dimensional case we have in zeroth order of approximation the Lagrangian for the field Ψ. In front of this Lagrangian we have the constant m. If this constant is positive, $m>0$, the theory avoids ghosts. Otherwise it possesses a particle with a negative kinetic energy. Thus, we have a condition

$$l^{[dc]}l_{[dc]}>3n(n-1) \tag{12.34}$$

and if (12.34) is satisfied, the theory is completely ghost-free. The condition (12.34) can be rewritten

$$l^{ab}(\varepsilon)l_{ba}<n(7-6n) \tag{12.34}$$

Using equations (7.54)-(7.58), one gets for $n>4$

$$\sum_{i=1}^{n}\frac{1-\mu\zeta_i}{1+\mu\zeta_i}<n(7-6n) \tag{12.34b}$$

This inequality can be easily satisfied. Let $\varepsilon < 0$ and let μ be such that for $i = i_0$ one has $1 + \mu\zeta_{i_0} = \varepsilon$. In this case one obtains

$$\sum_{\substack{i=1\\ i\neq i_0}}^{n} \frac{1-\mu\zeta_i}{1+\mu\zeta_i} + \frac{2}{\varepsilon} - 1 < n(7-6n)$$

If $|\varepsilon|$ is sufficiently small, our condition is satisfied and $1 + \mu\zeta_i \neq 0$. Moreover, we need μ to be a root of $\Phi(\mu)$.

Thus, this condition can be treated as a criterion for a gauge group choice. For example, for $G = SO(3)$ we get

$$m(SO(3)) = \frac{-2(36+7\mu^2)}{4+\mu^2} < 0$$

Thus, we should reject $SO(3)$. We can avoid tachyons if the masslike term is nonpositive, i.e.,

$$\frac{n+2}{2}\tilde{R}(\tilde{\Gamma}) \leq 0 \tag{12.35}$$

This is also a criterion for a gauge group G.

It is easy to see that the mass of the scalar field Ψ in a linear approximation is

$$M_{\Psi}^{\text{lin}} = \frac{n+2}{\sqrt{2}}\left(\frac{-\tilde{R}(\tilde{\Gamma})}{\mu k_{dc} l^{[dc]} - 3n(n-1)}\right)^{1/2} m_{\text{Pl}} \geq 0 \tag{12.36}$$

where

$$m_{\text{Pl}} = \left(\frac{c\hbar}{G_N}\right)^{1/2}$$

is the Planck mass. The total mass of the scalar field M_{Ψ}^{tot} is not equal to M_{Ψ}^{lin} (nonlinear interaction with remaining fields, i.e., gauge fields) and it can be found from the Yukawa behavior of the static, spherically symmetric solution of the full field equation, which we mentioned before. In Section 7 one finds that for large μ, $\tilde{R}(\tilde{\Gamma})$ behaves like

$$\tilde{R}(\tilde{\Gamma}) \sim \frac{C}{\mu} \tag{12.37}$$

where C is a constant. Thus, we get from equation (12.36)

$$M_{\Psi}^{\text{lin}} \sim \frac{M_{\text{Pl}}}{\sqrt{|\mu|}}\frac{n+2}{\sqrt{2}}\left(\frac{-C}{\mu k_{dc} l^{[dc]} - 3n(n-1)}\right)^{1/2} \tag{12.38}$$

According to (12.35), the constant C should be nonpositive, $C \leq 0$.

Finally, let us remark that in the linear approximation of the Lagrangian for the scalar field Ψ we can redefine

$$\Psi \to \frac{1}{\sqrt{|m|}}\Psi \tag{12.39}$$

In this way we will be dealing with the usual Lagrangian for a scalar field.

13. GEODETIC EQUATIONS IN LINEAR APPROXIMATION

Let us consider a generalized Kerner–Wong equation in the special case $\rho = 1$,

$$\frac{\bar{D}u^\alpha}{d\tau} - \frac{q^b}{m_0} l_{bd} g^{\alpha\beta} H^d{}_{\beta\gamma} u^\gamma + \frac{q^b}{m_0}(p_{bd} g^{[\alpha\beta]} + \mu k_{bd} g^{(\alpha\beta)}) L^d{}_{\beta\gamma} u^\gamma = 0 \tag{13.1}$$

$$\frac{dq^b}{d\tau} = 0$$

where q^b is the color (isotopic) charge of a test particle and m_0 its mass. Here, as in Section 4.10, we are using a different notation for the Killing–Cartan tensor, i.e., p_{ab} in place of h_{ab}. Using (12.6), one easily writes (13.1) up to the first order of approximation with respect to $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$,

$$\frac{\bar{D}u^\alpha}{d\tau} - \frac{q^b}{m_0} l_{bd}(\eta^{\alpha\beta} - \eta^{\alpha\sigma}\eta^{\beta\tau}h_{\tau\sigma}) H^d{}_{\beta\gamma} u^\gamma - \frac{q^b}{m_0}\mu k_{bd}\eta^{\alpha\beta} u^\gamma$$
$$\times [p^{de} l_{ec} H^c{}_{\beta\gamma} + (\delta^d{}_e - \mu^2 k^{dc} k_{ce})\eta^{\sigma\delta}(h_{[\beta\delta]} H^c{}_{\sigma\gamma} - h_{[\gamma\delta]} H^c{}_{\sigma\beta})] \tag{13.2}$$

where

$$k^{dc} = p^{da} p^{cb} k_{ab} \tag{13.3}$$

Let us consider the Kerner–Wong equation in the general case for $\Psi \neq 0$ (and not constant).

We have one more term

$$-\frac{\|q\|^2}{4m_0^2}\tilde{g}^{(\alpha\beta)} e^{2\Psi}\Psi_{,\beta} \quad \text{or} \quad -\frac{\|q\|^2}{8m_0^2}\tilde{g}^{(\alpha\beta)}\left(\frac{1}{\rho^2}\right)_{,\beta} \tag{13.4}$$

in terms of the field ρ, and

$$\|q\|^2 = (-p_{ab} q^b q^a) \tag{13.5}$$

$\|q\|^2$ is the norm of the color (isotopic) charge. One easily writes (13.4) up to the first order of approximation with respect to $h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$ and Ψ,

$$\frac{\bar{D}u^{\alpha}}{d\tau}-\frac{q^{b}}{m_{0}}l_{bd}(\eta^{\alpha\beta}-\eta^{\alpha\sigma}\eta^{\beta\tau}h_{\tau\sigma})H^{d}{}_{\beta\gamma}u^{\gamma}-\frac{q^{b}}{m_{0}}\mu k_{bd}\eta^{\alpha\beta}u^{\gamma}$$
$$\times[h^{de}l_{ec}H^{c}{}_{\beta\gamma}+(\delta^{d}{}_{e}-\mu^{2}k^{dc}k_{ce})\eta^{\sigma\delta}(h_{[\beta\delta]}H^{c}{}_{\sigma\gamma}-h_{[\gamma\delta]}H^{c}{}_{\sigma\beta})]$$
$$-\frac{\|q\|^{2}}{4m_{0}^{2}}\Psi_{,\beta}(\eta^{\alpha\beta}-\eta^{\alpha(\mu}\eta^{\nu)\beta}h_{\mu\nu})=0 \tag{13.6}$$

It is easy to see that the skewon field $h_{[\mu\nu]}$ has an influence on the motion of a test particle (13.2) and (13.6). The scalar field also has an influence on the motion in the linear approximation (13.6).

14. EQUATIONS OF MOTION FOR A TEST PARTICLE AND GEODETIC DEVIATION EQUATIONS

Let us come back to equation (4.13) and consider it for $\rho=1$ ($\Psi=0$). One gets

$$\frac{\bar{D}u^{\alpha}}{d\tau}+\frac{q^{c}}{m_{0}}\left[l_{cd}g^{\alpha\delta}H^{d}{}_{\beta\delta}-\frac{1}{2}(l_{cd}g^{\alpha\delta}-l_{dc}g^{\delta\alpha})L^{d}{}_{\beta\delta}\right]u^{\beta}=0 \tag{14.1}$$
$$\frac{q^{c}}{m_{0}}=\text{const}$$

Due to the compatibility condition (4.7) we have (see refs. 43 and 72) the first integral of motion

$$\gamma(u(\tau),u(\tau))=\gamma_{AB)}u^{A}(\tau)u^{B}(\tau)=\text{const} \tag{14.2}$$

or

$$g_{(\alpha\beta)}u^{\alpha}(\tau)u^{\beta}(\tau)+h_{ab}u^{a}u^{b}=\text{const} \tag{14.3}$$

Moreover, due to the second of equations (14.1), we have

$$h_{ab}u^{a}u^{b}=\text{const} \tag{14.4}$$

Thus, we get

$$\gamma(\text{hor}(u(\tau)),\text{hor}(u(\tau)))=g_{\alpha\beta}u^{\alpha}(\tau)u^{\beta}(\tau)=\text{const} \tag{14.5}$$

We consider only const ≥ 0, because otherwise we get unphysical worldlines.

Let us rewrite (14.1) in the following form:

$$m_{0}a^{\alpha}+q^{c}g^{\alpha\delta}l_{cd}u^{\beta}H^{d}{}_{\beta\delta}-\frac{q^{c}}{2}(l_{cd}g^{\alpha\delta}-l_{dc}g^{\delta\alpha})L^{d}{}_{\beta\delta}u^{\beta}=0 \tag{14.6}$$

where

$$2u^b = \frac{q^b}{m_0} \tag{14.7}$$

and

$$a^\alpha = \frac{\bar{D}u^\alpha}{d\tau} = \frac{\bar{D}}{d\tau}\left(\frac{dx^\alpha}{d\tau}\right) \tag{14.8}$$

is the covariant 4-acceleration of a test particle. Let us consider an initial Cauchy problem for (14.7) such that

$$\begin{aligned} x^\alpha(\tau_0) &= x_0^\alpha \\ \frac{dx^\alpha}{d\tau}(\tau_0) &= u_0^\alpha \\ g_{\alpha\beta} u_0^\alpha u_0^\beta &= 1 \end{aligned} \tag{14.9}$$

i.e., we consider timelike worldlines. Moreover, we can proceed in the same way with null lines. In this case we should put $m_0 = q^b = 0$ and u^b can have an interpretation as a coupling between a gauge field and a particle (i.e., a gluon).

Due to equation (14.5) we have for every $\tau \geq \tau_0$

$$g_{\alpha\beta} \frac{dx^\alpha}{d\tau}(\tau) \frac{dx^\beta}{d\tau}(\tau) = 1 \tag{14.10}$$

We give an interpretation of the additional term for the non-Abelian Lorentz-like force in equation (14.6), i.e.,

$$-\frac{q^c}{2}[l_{cd} g^{\alpha\delta} - l_{dc} g^{\delta\alpha}) L^d{}_{\beta\delta} u^\beta \tag{14.11}$$

To do this, let us consider equation (14.6) without this term, i.e.,

$$m_0 a^\alpha + q^c g^{\alpha\delta} l_{cd} u^\beta H^d{}_{\beta\delta} = 0 \tag{14.12}$$

This equation is a simple generalization of the equation of motion for a point charged particle (Kerner-Wong equation), known in symmetric non-Abelian Kaluza-Klein theory, to the nonsymmetric case. Now $g^{\alpha\gamma}$ is not symmetric and the covariant 4-acceleration is defined in terms of the connection $\tilde{\omega}^\alpha{}_\beta$ on E. This connection is of course compatible with the nonsymmetric metric $g_{\alpha\beta}$. One easily checks that

$$\frac{d}{d\tau}\left(g_{(\alpha\beta)} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}\right) = 2q^c g_{\gamma\alpha} g^{\alpha\delta} l_{cd} H^d{}_{\delta\beta}\left(\frac{dx^\beta}{d\tau}\right)\left(\frac{dx^\alpha}{d\tau}\right) \neq 0 \tag{14.13}$$

Thus, in general, equation (4.12) does not have the first integral of motion (14.5). This means that we are unable in general to preserve the initial normalization for the 4-velocity of a test particle. If we want to have the normalization (14.10), we should add to (14.12) the auxiliary condition

$$\Phi(u^{\alpha}) = 0 \tag{14.14}$$

where

$$\Phi(u^{\alpha}) = g_{(\alpha\beta)} u^{\alpha} u^{\beta} - 1 \tag{14.15}$$

For a null line we have

$$\Phi(u^{\alpha}) = g_{\alpha\beta} u^{\alpha} u^{\beta} = 0$$

and

$$a^{\alpha} + u^{c} g^{\alpha\delta} l_{cd} u^{\beta} H^{d}{}_{\beta\delta} = 0$$

The auxiliary condition (14.15) is a nonholonomic constraint. This constraint is nonintegrable and nonlinear (quadratic in velocities). According to the general theory of mechanical systems with constraints, we know that in such systems we have the so-called reaction forces of constraints. Thus, we should write equation (14.12) in the following form:

$$m_0 a^{\alpha} = -2u^{c} g^{\alpha\delta} l_{cd} u^{\beta} H^{d}{}_{\beta\delta} + Q^{\alpha} \tag{14.16}$$

$$\Phi(u^{\alpha}) = g_{\alpha\beta} u^{\alpha} u^{\beta} - 1 = 0 \tag{14.17}$$

Q^{α} is a reaction force of the constraint (14.17). The force Q^{α} must be such that (14.17) is automatically satisfied during a motion. Let us find this force. In order to do this, let us multiply both sides of (14.16) by $g_{(\alpha\beta)} u^{\beta}$ and integrate from τ_0 to τ. One gets

$$\begin{aligned} \tfrac{1}{2} m_0 \Phi(u^{\alpha}) &= \tfrac{1}{2} m_0 (g_{(\alpha\beta)} u^{\alpha} u^{\beta} - 1) \\ &= \int_{\tau_0}^{\tau} (g_{(\alpha\beta)} u^{\beta} Q^{\alpha} - 2m_0 u^{c} u^{\gamma} g_{(\alpha\beta)} g^{\alpha\delta} l_{cd} u^{\beta} H^{d}{}_{\gamma\delta})\, d\tau = 0 \end{aligned} \tag{14.18}$$

For (4.17) satisfied, we get

$$\int_{\tau_0}^{\tau} (g_{(\alpha\beta)} u^{\beta} Q^{\alpha} - 2m_0 u^{c} u^{\gamma} g_{(\alpha\beta)} g^{\alpha\delta} l_{cd} u^{\beta} H^{d}{}_{\gamma\delta})\, d\tau = 0 \tag{14.19}$$

Moreover, equation (14.19) is satisfied for every $\tau \geq \tau_0$. Thus, we get

$$g_{(\alpha\beta)} u^{\beta} Q^{\alpha} - 2m_0 u^{c} u^{\gamma} g_{(\alpha\beta)} g^{\alpha\delta} l_{cd} u^{\beta} H^{d}{}_{\gamma\delta} = 0 \tag{14.20}$$

It is easy to see that equation (14.20) has the solution

$$Q^{\alpha}=2m_0 l_{cd}u^c u^{\gamma} g^{\alpha\delta} H^d{}_{\gamma\delta} \tag{14.21}$$

If we put (14.21) into (14.16), we get

$$m_0 a^{\alpha}=0 \tag{14.22}$$

This solution has a simple physical interpretation. Equation (14.22) is an equation of motion for an uncharged test particle. There is no Lorentz force. It corresponds to a choice $u^b=0$ or equivalently $q^b=0$. Let us come back to equation (14.20) and transform it using condition (4.10). One gets

$$\begin{aligned}&\tfrac{1}{2}g_{\alpha\beta}u^{\beta}Q^{\alpha}+\tfrac{1}{2}g_{\beta\alpha}u^{\beta}Q^{\alpha}-m_0 u^c u^{\gamma}u^{\beta}L^d{}_{\gamma\delta}\\ &\quad\times(l_{cd}g_{\beta\alpha}g^{\alpha\delta}-l_{dc}g_{\alpha\beta}g^{\delta\alpha})=0\end{aligned} \tag{14.23}$$

Equation (14.23) has the solution

$$Q^{\alpha}=-\frac{q^c}{2}[l_{cd}g^{\alpha\delta}-l_{dc}g^{\delta\alpha})L^d{}_{\beta\delta}u^{\beta} \tag{14.24}$$

Thus, equation (14.24) gives us an interpretation for an additional term for a non-Abelian Lorentz force in equation (14.1) or equation (14.6). This additional term is a reaction force of the nonintegrable, nonholonomic, nonlinear constraints. It is easy to see that our constraints are nonideal, for Q^{α} is not proportional to a gradient of Φ. The constraints seem to be similar to the so-called servo-constraints. For a nonholonomic (nonintegrable) constraint we have the following statement: A variational problem with differential (nonintegrable, nonholonomic) constraints cannot be reduced to a form where the variation of a certain quantity (an action) is put equal to zero. This is true in the much simpler case of linear nonholonomic constraints (ref. 73). Thus, unfortunately, we cannot formulate a principle of minimal action for equation (14.1). Moreover, we are still able to interpret the additional term in the non-Abelian Lorentz force as a reaction force of the nonholonomic constraints (14.17). Moreover, the force Q^{α} is absorbed by a geometry (it is geometrized). For a null line we proceed similarly. However, one can try to formulate a local Gauss-like principle in order to derive equation (14.1). Thus, let us consider a local Gauss-like principle for this equation,

$$\delta Z^2=0$$

modulo constraints (14.15), where

$$Z^2=\frac{m_0}{2}\tilde{g}^{(\alpha\beta)}\bar{f}_{\alpha\gamma}\left(a^{\gamma}-\frac{F^{\gamma}}{m_0}\right)\bar{f}_{\beta\mu}\left(a^{\mu}-\frac{F^{\mu}}{m_0}\right)$$

The matrix $\bar{f}_{\alpha\beta}$ is defined as follows:

$$\bar{f}^{\zeta}{}_{\rho}\bar{f}^{\rho}{}_{\delta} = \frac{q^c}{2m_0}(l_{cd}g^{\zeta\delta} - g^{\delta\zeta}l_{dc})L^d{}_{\delta\beta} = b^{\zeta}{}_{\delta}$$

$$\bar{f}^{\zeta}{}_{\rho} = g_{(\rho\alpha)}\bar{f}^{\zeta\alpha}, \qquad \bar{f}^{\alpha\beta}\cdot\bar{f}_{\alpha\gamma} = \delta^{\beta}{}_{\gamma}, \qquad \det(\bar{f}_{\alpha\beta}) \neq 0$$

Thus, $\bar{f}$ exists only if the matrix $b^{\alpha}{}_{\beta}$ is invertible and positively defined. In this case we are able to formulate a Gauss-like local principle for (14.1). Taking the variation of Z^2 and constraints with respect to a^{α} (a covariant acceleration with respect to the connection $\bar{\Gamma}^{\alpha}{}_{\beta\gamma}$ on E), one gets

$$\bar{f}^{\mu\zeta}\bar{f}_{\mu\gamma}(m_0 a^{\gamma} - F^{\gamma}) + 2r\bar{f}^{\mu\zeta}g_{(\rho\mu)}\bar{f}^{\rho\nu}g_{(\nu\delta)}u^{\delta} = 0$$

where r is a Lagrange multipler. Using the definition of the matrix $\bar{f}_{\alpha\beta}$, we get equation (14.1) and $r = -\frac{1}{2}$.

Let us calculate an acceleration function (an analogue of an acceleration energy) in this case. One gets $S = \frac{1}{2}m_0\tilde{g}^{(\alpha\beta)}\bar{f}_{\alpha\gamma}\bar{f}_{\beta\mu}a^{\gamma}a^{\mu}$. The other form of our equation is

$$\frac{\partial S}{\partial a^{\alpha}} = \tilde{F}_{\alpha} + R_{\alpha} = g_{(\alpha\beta)}(\tilde{F}^{\beta} + R^{\beta})$$

where R_{α} is an ideal reaction force and $\tilde{F}_{\alpha}$ is an external force, and

$$\tilde{F}^{\alpha} = \bar{f}^{\mu\alpha}\bar{f}_{\mu\gamma}F^{\gamma} = q^c\bar{f}^{\mu\beta}\bar{f}_{\mu\gamma}l_{cd}g^{\alpha\delta}H^d{}_{\beta\delta}u^{\beta}$$

Let us find the form of $S^{\mu\alpha}$, which is an inverse tensor of $S_{\alpha\beta}$. One gets

$$S^{\alpha\gamma} = \frac{m_0}{2}g_{(\delta\mu)}\bar{f}^{\delta\alpha}\bar{f}^{\mu\gamma} = \frac{q^c}{2}\cdot\tilde{g}^{(\delta\alpha)}(l_{cd}g^{\gamma\beta} - l_{dc}g^{\beta\gamma})\cdot L^d{}_{\beta\delta} = S^{\gamma\alpha}$$

where $S^{\alpha\gamma}S_{\alpha\mu} = \delta^{\gamma}{}_{\mu}$. Thus, we can reformulate a satisfactory condition of the application of our Gauss-like principle:

1. $\det(S^{\gamma\alpha}) \neq 0$.
2. $S^{\gamma\alpha} = S^{\alpha\gamma}$.
3. S is positively defined.

Note that sometimes in theoretical mechanics one gets in the case of nonlinear, nonholonomic constraints a wrong equation from the Gauss principle, *even* though this principle is *applicable* for nonlinear, nonholonomic constraints. We mean here a well-known example given by Appel and Hamel. Thus, the formulation of a Gauss-like local principle with nonideal reaction forces seems to be rather unexpected, even under some conditions. Let us recall that this example is connected to a motion in an extremal situation (a parameter $\bar{p} \to 0$) and the constraints are quadratic in velocities (see ref. 123). The model with nonlinear constraints for the

Appel–Hamel system is incorrect. Thus, we should be really very satisfied that in our case we get correct results (a correct equation of motion), even the reaction force is nonideal. Summing up, we conclude that we are able to get an equation of motion for a test particle in N^2AK^2T from a Gauss-like principle with nonlinear constraints, which is not commonly possible even in theoretical mechanics. We rewrite the satisfactory conditions for the applications of the Gauss-like local principle for equation (14.1):

1. The matrix $b_{\alpha\delta}$ is positively defined and invertible, $\det(b_{\alpha\beta}) \neq 0$, where

$$b_{\alpha\beta} = \frac{q^c}{2}\left(\mu k_{cd}L^d{}_{\alpha\beta} + \frac{1}{2}\,(l_{cd}g^{\zeta\delta}g_{\alpha\zeta} - l_{dc}g^{\delta\zeta}g_{\zeta\alpha})_{L^d{}_{\delta\beta}}\right)$$

2. $b_{\alpha\beta} = b_{\beta\alpha}$.

The force F^α is as follows:

$$F^\alpha = q^c g^{\alpha\delta} l_{cd} H^d{}_{\beta\delta} u^\beta$$

During a motion the quantity Z^2 is minimalized *modulo* nonlinear, nonholonomic constraints (14.17).

The constraints are nonideal and the force Q^α is a nonideal reaction force. The nonideal reaction force can be expressed by the ideal reaction force $R^\alpha = r\,\partial\Phi/\partial\mu_\alpha = ru^\alpha$, $r \neq 0$,

$$Q^\alpha = \psi P^\alpha{}_\beta R^\beta, \qquad \psi \neq 0$$

$$P^\alpha{}_\beta = \frac{q^c}{2}\,[l_{cd}g^{\alpha\delta} - l_{dc}g^{\delta\alpha})_{L^d{}_{\beta\delta}}$$

Let us consider the more general case of geodetic equations where $\rho \neq \text{const}$. We have

$$\frac{\bar{D}u^\alpha}{d\tau} + \left(\frac{q^c}{m_0}\right)u^\beta\left[l_{cd}g^{\alpha\delta}H^d{}_{\beta\delta} - \frac{1}{2}\,(l_{cd}g^{\alpha\delta} - l_{dc}g^{\delta\alpha})L^d{}_{\beta\delta}\right]$$

$$-\frac{\|q\|^2}{8m_0^2}\,\tilde{g}^{(\alpha\beta)}\left(\frac{1}{\rho^2}\right)_{,\beta} = 0 \tag{14.25}$$

$$\frac{q^c}{m_0} = \text{const} \tag{14.25}$$

where $\|q\| = (-h_{ab}q^a q^b)^{1/2}$ is a length of color (isotopic) charge in the Lie algebra $\mathfrak{g}$ of the group G.

Let us find similarly as in the electromagnetic case the physical interpretation of the additional term

$$\frac{1}{8}\,\frac{\|q\|^2}{m_0^2}\,\tilde{g}^{(\beta\alpha)}\left(\frac{1}{\rho^2}\right)_{,\beta} \tag{14.27}$$

This term describes the scalar, velocity-indpendent force acting on the test particle. The force depends on the "chemical composition" of the particle, because it has in front the factor $(\|q\|/m_0)^2$. Thus, it could be considered as a new type of force, maybe the "fifth force" (see refs. 54, 55, and 96–102). Let us multiply both sides of equation (14.25) by $g_{(\alpha\gamma)}u$ in order to understand the effect of an action of the scalar force on the test particle motion. One easily gets

$$\frac{d}{d\tau}\left(m_0 g_{(\alpha\beta)}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right)=\frac{1}{8}\frac{\|q\|^2}{m_0}\frac{d}{d\tau}\left(\frac{1}{\rho^2}\right) \tag{14.28}$$

where

$$u^\alpha=\frac{dx^\alpha}{d\tau}$$

and

$$\frac{d}{d\tau}\left(\frac{1}{\rho^2}\right)=\left(\frac{1}{\rho^2}\right)_{,\beta}u^\beta$$

or

$$g_{(\alpha\beta)}u^\alpha u^\beta-\frac{\|q\|^2}{8{m_0}^2\rho^2}=\text{const} \tag{14.28a}$$

which is a first integral of motion.

Let us suppose that $H^a{}_{\mu\nu}=0$ and (of course) $L^a{}_{\mu\nu}=0$. It is very well known that

$$m_0 g_{(\alpha\beta)}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=E_p \tag{14.29}$$

has been considered the energy of a test particle in a rest frame. Thus, the scalar force is changing the rest energy of a test particle in the following way:

$$\frac{dE_p}{d\tau}=\frac{\|q\|^2}{8m_0}\frac{d}{d\tau}\left(\frac{1}{\rho^2}\right) \tag{14.30}$$

Equation (14.30) gives the first integral of motion

$$E_p-\frac{\|q\|^2}{8m_0\rho^2}=\text{const} \tag{14.31}$$

Thus, the energy of a single test particle is changing during its motion according to (14.31). This result is easily understandable because of the physical interpretation of the field ρ. This field is connected to the effective gravitational constant

$$G_{\text{eff}}=G_N\rho^{(n+2)} \tag{14.32}$$

(G_N is the Newton constant). It means that if $\rho \neq$ const the effective strength of the gravitational interaction is changing during a motion and because of this the field ρ changes the rest energy of the test particle. Moreover, the total energy of a test particle in a rest frame and the field ρ is constant.

In general the scalar force can act as a "friction force" or "amplification force" transforming the energy of a particle into the energy of the field ρ and vice versa. If the field ρ depends only on time, then equation (14.31) describes the change of the energy of a test particle due to the time dependence of the effective strength of gravitational interactions, in a composition-dependent way.

Let us solve equation (6.2) in a weak-field approximation using an iterative method. In order to do this, we write equation (6.2) in a more convenient form,

$$L^b{}_{\beta\alpha} = h^{bc}(l_{cd}g_{\alpha\mu}g^{\mu\gamma}H^d{}_{\beta\gamma} - l_{cd}g_{[\alpha\mu]}g^{\mu\gamma}L^d{}_{\beta\gamma} - l_{dc}g_{[\mu\beta]}g^{\gamma\mu}L^d{}_{\gamma\alpha}) \qquad (14.33)$$

and define the transformation

$$\overset{(n+1)}{L^b}{}_{\beta\alpha} = M^b{}_e{}^{\mu\nu}{}_{\beta\alpha}\overset{(n)}{L^e}{}_{\mu\nu} \qquad (14.34)$$

such that

$$\overset{(0)}{L^b}{}_{\beta\alpha} = h^{bc}l_{cd}H^d{}_{\beta\alpha} \qquad (14.35)$$

$$\overset{(n+1)}{L^b}{}_{\beta\alpha} = h^{bc}(l_{cd}g_{\alpha\mu}g^{\mu\gamma}H^d{}_{\beta\gamma} - l_{cd}g_{[\alpha\mu]}g^{\mu\gamma}\overset{(n)}{L^d}{}_{\beta\gamma} - l_{dc}g_{[\mu\beta]}g^{\gamma\mu}\overset{(n)}{L^d}{}_{\gamma\alpha}) \qquad (14.36)$$

One easily gets

$$\overset{(n+1)}{L^d}{}_{\beta\alpha} = (M^{n+1})^d{}_e{}^{\mu\nu}{}_{\beta\alpha}\overset{(0)}{L^e}{}_{\mu\nu} = (M^{n+1})^d{}_e{}^{\mu\nu}{}_{\beta\alpha}h^{ef}l_{fd}H^d{}_{\mu\nu} \qquad (14.37)$$

The power $(n+1)$ means the $(n+1)$-iteration of the transformation (14.36). We get

$$\overset{(n+1)}{L^d}{}_{\beta\alpha} - \overset{(n)}{L^d}{}_{\beta\alpha} = -h^{bc}[l_{cd}g_{[\alpha\mu]}g^{\mu\gamma}(\overset{(n)}{L^d}{}_{\beta\gamma} - \overset{(n-1)}{L^d}{}_{\beta\gamma})$$
$$+ l_{dc}g_{[\mu\beta]}g^{\gamma\mu}(\overset{(n)}{L^d}{}_{\beta\gamma} - \overset{(n-1)}{L^d}{}_{\beta\gamma})] \qquad (14.38)$$

Now let us suppose that the field $g_{\alpha\beta}$ is weak. This means that we assume that

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \qquad (14.39)$$

$$g^{\alpha\beta} = \eta^{\alpha\beta} + \tilde{h}^{\alpha\beta} \tag{14.40}$$

$$|h_{\alpha\beta}|, |\tilde{h}^{\alpha\beta}| < \alpha \ll 1$$

where $\eta_{\alpha\beta}$ is the Minkowski tensor.

In this case one gets

$$g^{\alpha\beta} \cong \eta^{\alpha\beta} - \eta^{\sigma\alpha}\eta^{\gamma\beta}h_{\gamma\sigma} \tag{14.41}$$

The skew-symmetric tensors $L^d{}_{\mu\nu}$ form a natural $6n$-dimensional vector space. Let us define the following norm in this space:

$$\|L\| = \max_{\substack{\mu,\nu=1,2,3,4\\ d=1,2,\dots,n}} |L^d{}_{\mu\nu}| \tag{14.42}$$

Thus, our space becomes a Banach space. For sufficiently small α one finds

$$\|\overset{(n+1)}{L} - \overset{(n)}{L}\| \le \beta(\alpha)\|\overset{(n)}{L} - \overset{(n-1)}{L}\| \tag{14.43}$$

where

$$0 < \beta(\alpha) = 96n^2\tilde{h}(h + |\mu|k)\alpha < 1$$

if

$$\alpha < \frac{1}{96n^2\tilde{h}(h+|\mu|k)}$$

and

$$h = \max_{a,b}(|h_{ab}|), \qquad \tilde{h} = \max_{a,b}(|\tilde{h}_{ab}|), \qquad k = \max_{a,b}(|k_{ab}|)$$

Equation (14.41) means that the transformation is a contraction. According to the Banach theorem, this transformation has a fix point

$$\overset{(\infty)}{L}{}^d{}_{\beta\alpha} = M^d{}_e{}^{\mu\nu}{}_{\beta\alpha}\overset{(\infty)}{L}{}^e{}_{\mu\nu} \tag{14.44}$$

such that

$$\overset{(\infty)}{L}{}^d{}_{\beta\alpha} = \lim_{n=\infty}(M^n)^d{}_e{}^{\mu\nu}{}_{\beta\alpha}h^{ec}l_{cd}H^d{}_{\mu\nu}$$

$$= \overset{(\infty)}{M}{}^d{}_e{}^{\mu\nu}{}_{\beta\alpha}h^{ec}l_{cd}H^d{}_{\mu\nu} \tag{14.45}$$

The limit is understood in the sense of the norm (14.40) and

$$\overset{(\infty)}{M}{}^{d}{}_{e}{}^{\mu\nu}{}_{\beta\alpha} = \lim_{n=\infty}(M^{n})^{d}{}_{e}{}^{\mu\nu}{}_{\beta\alpha} \tag{14.46}$$

The limit (14.46) is understood in the sense of the usual linear operator topology generated by a topology of a Banach space. According to the Banach theorem, there is one and only one fix point of the transformation (14.36) (in a weak-field approximation). Thus, we get that

$$L^{d}{}_{\beta\alpha} = \overset{(\infty)}{M}{}^{d}{}_{e}{}^{\mu\nu}{}_{\beta\alpha}\, h^{ec} l_{cf} H^{f}{}_{\mu\nu} \tag{14.47}$$

In this way we can rewrite the equations of motion for a test particle in the following way:

$$\frac{\bar{D}u^{\alpha}}{d\tau} + \left(\frac{q^{c}}{m_0}\right) u^{\beta}\left[l_{cd} g^{\alpha\delta} H^{d}{}_{\beta\delta} - \frac{1}{2}(l_{cd} g^{\alpha\delta} - l_{dc} g^{\delta\alpha}) \times \overset{(\infty)}{M}{}^{d}{}_{e}{}^{\mu\nu}{}_{\beta\delta} h^{ec} l_{cf} H^{f}{}_{\mu\nu}\right] - \frac{\|q\|^2}{8m_0^2}\, \tilde{g}^{(\alpha\beta)}\left(\frac{1}{\rho^2}\right)_{,\beta} = 0 \tag{14.48}$$

Let us remark that, as in the electromagnetic case, we can consider different equations of motion for a test particle, i.e.,

$$\left(\frac{d^2x^{\alpha}}{d\tau^2} + \left\{ {\alpha \atop \beta\ \ \gamma} \right\} \frac{dx^{\beta}}{d\tau}\frac{dx^{\gamma}}{d\tau}\right) + \left(\frac{q^{c}}{m_0}\right)\left(\frac{dx^{\beta}}{d\tau}\right) \times \left[l_{cd} g^{\alpha\delta} H^{d}{}_{\beta\delta} - \frac{1}{2}(l_{cd} g^{\alpha\delta} - l_{dc} g^{\delta\alpha}) L^{d}{}_{\beta\delta}\right] - \frac{\|q\|^2}{8m_0^2}\, \tilde{g}^{(\alpha\beta)}\left(\frac{1}{\rho^2}\right)_{,\beta} = 0 \tag{14.49}$$

and

$$\left(\frac{q^{c}}{m_0}\right) = \text{const} \tag{14.49a}$$

Let us notice that equation (14.49) has the same integral of motion as (14.25), i.e.,

$$g_{(\alpha\beta)}\left(\frac{dx^{\alpha}}{d\tau}\right)\left(\frac{dx^{\beta}}{d\tau}\right) - \frac{\|q\|^2}{8m_0^2\rho^2} = \text{const} \tag{14.50}$$

In the case of $\rho = \text{const}$ we get

$$g_{(\alpha\beta)}\left(\frac{dx^\alpha}{d\tau}\right)\left(\frac{dx^\beta}{d\tau}\right) = \text{const} \tag{14.51}$$

Thus, we formulate the following theorem:

Theorem IV:

1. Let conditions 1-3 from Theorem III be satisfied (see Section 4.3).
2. Let $\rho = \text{const}$.

Then Theorem III is satisfied and one has the first integral of motion of geodetic equations with respect to the connections $\omega^A{}_B$ and $\check{\omega}^A{}_B$,

$$\gamma(\text{hor}(u(\tau)), \text{hor}(u(\tau))) = \text{const}$$

Equations (14.49) and (14.49a) are geodetic equations with respect to the connection $\check{\omega}^A{}_B$ defined on $\underline{P}$ such that in place of the connection $\bar{\omega}^\alpha{}_\beta$ we put in (6.1)

$$\check{\omega}^\alpha{}_\beta = \left\{ {}_\beta{}^\alpha{}_\gamma \right\} \bar{\theta}^\gamma \tag{14.52}$$

where $\{{}_\beta{}^\alpha{}_\gamma\}$ is a Christoffel symbol formed for a metric $g_{(\alpha\beta)}$. We can also consider different Christoffel symbols formed for a metric $p_{\alpha\beta} = p_{\beta\alpha}$, where

$$p_{\alpha\beta} g^{(\alpha\gamma)} = \delta_\beta{}^\gamma$$

and $g^{\alpha\beta}$ is an inverse tensor for $g_{\alpha\beta}$.

Let us consider a geodesic deviation equation in our theory,

$$u^B \nabla_B v^A - [\nabla_B, \nabla_B] u^A u^B \zeta^M = 0 \tag{14.53}$$

or

$$u^B \nabla_B v^A + R^A{}_{CMB} u^C \zeta^M u^B - Q^N{}_{MB} \nabla_N u^A \zeta^M u^B = 0 \tag{14.54}$$

and

$$U^B \nabla_B u^A = 0 \tag{14.55}$$

where $u^A = u^A = dx^A/d\tau$, $v^A = d\zeta^A/d\tau$.

In this way we consider a generalization of the geodetic deviation equations to the $(n+4)$-dimensional case in a non-Riemannian geometry. Using equations (6.1), (6.11)-(6.16), and (6.22a)-(6.22l), one gets from equation (14.54):

$$(u^\beta \bar{\nabla}_\beta v^\alpha + \bar{R}^\alpha{}_{\beta\mu\nu} u^\beta \zeta^\mu u^\nu - \bar{Q}^\nu{}_{\mu\beta}(\bar{\Gamma}) \bar{\nabla}_\nu u^\alpha \zeta^\mu u^\beta)$$

$$
\begin{aligned}
&+\left(\frac{q^b}{2m_0}\right) l_{bd} g^{\alpha\beta}(2H^d{}_{\nu\beta}-L^d{}_{\nu\beta})v^\nu-\rho^2 v^n l_{dn} g^{\delta\alpha}L^d{}_{\delta\beta}u^\beta\\
&-\left(\frac{q^n}{2\rho m_0}\right)\tilde{g}^{(\alpha\beta)}\rho_{,\beta} l_{bn} v^b\\
&+2\rho^2(l_{cd}g^{\alpha\omega}(2H^d{}_{\omega[\mu}-L^d{}_{\omega[\mu})L^c{}_{\nu]\beta}+l_{db}g^{\delta\alpha}L^d{}_{\delta\alpha}H^b{}_{\mu\nu})u^\gamma\zeta^\mu u^\nu\\
&+\frac{q^b}{2m_0\rho^2}\{2\bar{\nabla}_{[\mu}(\rho^2 l_{db}g^{\alpha\beta}(2H^d{}_{\nu]\beta}-L^d{}_{\nu]\beta}))\\
&+\rho^2 l_{bd}g^{\alpha\beta}(2H^d{}_{\gamma\beta}-L^d{}_{\gamma\beta})\bar{Q}^\gamma{}_{\mu\nu}(\bar{\Gamma})\\
&-\rho\tilde{g}^{(\alpha\beta)}\rho_{,\beta}l_{bc}H^c{}_{\mu\nu}+2\rho l_{bd}g^{\alpha\beta}(2H^d{}_{[\mu|\beta|}-L^d{}_{[\mu|\beta|}))g_{|\delta|\nu]}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\}\zeta^\mu u^\nu\\
&+\{\rho l_{bd}g^{\alpha\omega}g_{\beta\delta}\tilde{g}^{(\delta\zeta)}\rho_{,\zeta}(2H^d{}_{\mu\omega}-L^d{}_{\mu\omega})-\bar{\nabla}_\mu(\rho^2 l_{db}g^{\delta\alpha}L^d{}_{\delta\beta})\\
&-\rho\tilde{g}^{(\alpha\omega)}\rho_{,\omega}l_{cd}L^c{}_{\beta\mu}\}\\
&\times u^\beta\left(\zeta^\mu\frac{q^b}{2\rho m_0}-\zeta^b u^\mu\right)+\left(2\rho^4 l_{d[b}l_{|e|f]}g^{\delta\alpha}g^{\zeta\gamma}L^d{}_{\delta\gamma}L^e{}_{\zeta\beta}\right.\\
&\left.+\tilde{g}^{(\alpha\omega)}\rho_{,\omega}g_{\beta\delta}\tilde{g}^{(\delta\zeta)}\rho_{,\zeta}l_{[bf]}+\frac{\rho^2}{2}l_{dp}g^{\delta\alpha}L^d{}_{\delta\beta}C^p{}_{bf}\right)u^\beta\left(\frac{\zeta^b q^f}{2m_0\rho^2}\right)\\
&+\{\bar{\nabla}_\mu(\rho\tilde{g}^{(\alpha\beta)}\rho_{,\beta})l_{ba}+\tilde{\bar{\nabla}}_a(\zeta^2 l_{bd}(2H^d{}_{\mu\beta}-L^d{}_{\mu\beta}))\\
&-\rho^4 l_{da}l_{bf}g^{\delta\alpha}g^{\gamma\beta}L^d{}_{\delta\gamma}(2H^f{}_{\mu\beta}-L^f{}_{\mu\beta})-\tilde{g}^{(\alpha\zeta)}\rho_{,\zeta}g_{\delta\mu}\tilde{g}^{(\delta\nu)}\rho_{,\nu}l_{ba}\}\left(\frac{q^b}{2m_0\rho^2}\right)\\
&\times\left(\zeta^a u^\mu-\zeta^\mu\frac{q^a}{2m_0}\right)+(\rho\tilde{g}^{(\alpha\beta)}\rho_{,\beta}l_{bk}C^k{}_{ac}+2\rho^3\tilde{g}^{(\gamma\beta)}\rho_{,\beta}g^{\delta\alpha}L^d{}_{\delta\gamma}l_{d[a}l_{|b|c]})\\
&\times\left(\frac{q^b\zeta^a q^c}{4\rho^4 m_0^2}\right)-(2\rho^2 l_{bd}g^{\nu\beta}H^d{}_{\gamma\beta}+\rho^2(l_{bd}g^{\nu\beta}+l_{db}g^{\beta\nu})L^d{}_{\beta\gamma})\\
&\times\left(\nabla_\nu u^\alpha+\frac{1}{2}\left(l_{fd}g^{\alpha\beta}(2H^d{}_{\nu\beta}-L^d{}_{\nu\beta})\left(\frac{q^f}{m_0}\right)\right)\right)\left(v^\gamma\frac{q^b}{2\rho^2 m_0}-v^b u^\gamma\right)\\
&-2\mu\rho\tilde{g}^{(\nu\beta)}\rho_{,\beta}k_{bc}\left(\bar{\nabla}_v u^\alpha+\frac{1}{2}\left(l_{bf}g^{\alpha\gamma}(2H^d{}_{\nu\gamma}-L^d{}_{\nu\gamma})\left(\frac{q^f}{m_0}\right)\right)\right)\left(\frac{q^c v^b}{2m_0\rho}\right)\\
&+2(H^n{}_{\mu\nu}-L^n{}_{\mu\nu})\left(\rho^2 l_{dn}g^{\delta\alpha}L^d{}_{\delta\beta}u^\beta-\frac{1}{2\rho}\tilde{g}^{(\alpha\beta)}\rho_{,\beta}l_{bn}\left(\frac{q^b}{m_0}\right)\right)v^\mu u^\nu\\
&-g_{\zeta\mu}\tilde{g}^{(\alpha\beta)}\rho_{,\gamma}\left(\rho l_{dm}g^{\delta\alpha}L^d{}_{\delta\beta}u^\beta+\frac{1}{2\rho^2}\tilde{g}^{(\alpha\beta)}\rho_{,\beta}l_{bm}\left(\frac{q^b}{m_0}\right)\right)\\
&\times\left(v^\mu\left(\frac{q^b}{2\rho^2 m_0}\right)-v^m u^\mu\right)-\frac{1}{2\rho^2}\tilde{Q}^n{}_{mb}(\tilde{\Gamma})\left(\rho^2 l_{dn}g^{\delta\alpha}L^d{}_{\delta\beta}u^\beta-\frac{1}{2\rho}\tilde{g}^{(\alpha\beta)}\rho_{,\beta}\right.\\
&\left.\times l_{pn}\left(\frac{q^p}{m_0}\right)\right)v^m\left(\frac{q^b}{m_0}\right)=0 \qquad (14.56)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{dv^a}{d\tau}+\tilde{R}^a{}_{bcd}\left(\frac{q^b}{2\rho^2 m_0}\right)\zeta^c\left(\frac{q^d}{2\rho^2 m_0}\right)-\tilde{Q}^n{}_{mb}(\tilde{\Gamma})\tilde{\nabla}_n\left(\frac{q^a}{2\rho^2 m_0}\right)v^m\left(\frac{q^b}{2\rho^2 m_0}\right)\\
&\quad+\Bigg(2\rho^2 l_{bd}g^{\gamma\delta}(2H^d{}_{[\nu|\delta|}-L^d{}_{[\nu|\delta|})L^a{}_{|\gamma|\mu]}+\frac{1}{\rho}\,\delta^a{}_b g_{\delta\alpha}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\bar{Q}^\alpha{}_{\mu\nu}(\bar{\Gamma})\\
&\quad-2\delta^a{}_b\bar{\nabla}_{[\nu}\left(\frac{1}{\rho}\,g_{|\delta|\mu]}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\right)\Bigg)\left(\frac{q^b\zeta^\mu U^\nu}{2\rho^2 m_0}\right)\\
&\quad+(-\rho\tilde{g}^{(\gamma\beta)}\rho_{,\beta}l_{bc}L^a{}_{\gamma\beta}-\rho l_{bd}\tilde{g}^{(\beta\nu)}\rho_{,\nu}(2H^d{}_{\mu\beta}-L^d{}_{\mu\beta})\delta^a{}_c)\\
&\quad\times\left(\frac{q^b}{2\rho^2 m_0}\right)\left(\frac{\zeta^\mu q^c}{2\rho^2 m_0}-\zeta^c u^\mu\right)+\Bigg(\tilde{\nabla}_b L^a{}_{\beta\gamma}-\bar{\nabla}_\gamma\left(\frac{1}{\rho}\,g_{\beta\delta}\tilde{g}^{(\delta\mu)}\rho_{,\mu}\right)\delta^a{}_b\\
&\quad+\rho^2 l_{db}g^{\delta\mu}L^a{}_{\mu\gamma}L^d{}_{\delta\beta}-\frac{1}{\rho^2}\,g_{\delta\gamma}\tilde{g}^{(\delta\beta)}\rho_{,\beta}g_{\beta\nu}\tilde{g}^{(\nu\mu)}\rho_{,\mu}\delta^a{}_b\Bigg)\\
&\quad\times u^\beta\left(\zeta^b u^\gamma-\zeta^\gamma\left(\frac{q^b}{2\rho^2 m_0}\right)\right)\\
&\quad+\left(-\frac{1}{\rho}\,g_{\beta\delta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}C^a{}_{bc}+2\rho g_{\gamma\nu}\tilde{g}^{(\nu\mu)}\rho_{,\mu}g^{\delta\gamma}L^d{}_{\delta\beta}l_{d[b}\delta^a{}_{c]}\right)u^\beta\left(\frac{\zeta^b q^c}{4\rho^4 m_0^2}\right)\\
&\quad-2g_{\gamma\delta}\tilde{g}^{(\delta\nu)}\rho_{,\nu}\tilde{g}^{(\gamma\beta)}\rho_{,\gamma}l_{b[c}\delta^a{}_{d]}\left(\frac{q^b\zeta^d q^c}{4\rho^4 m_0^2}\right)\\
&\quad+\Bigg(2\bar{\nabla}_{[\mu}L^a{}_{|\beta|\nu]}+L^a{}_{\beta\gamma}\bar{Q}^\gamma{}_{\mu\nu}(\bar{\Gamma})+\frac{2}{\rho}\,g_{\beta\delta}\tilde{g}^{(\delta\gamma)}{}_{,\gamma}H^a{}_{\mu\nu}\\
&\quad+\frac{2}{\rho}\,g_{\gamma[\mu}L^a{}_{|\beta|\nu}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\Bigg)u^\beta\zeta^\mu u^\nu-\frac{1}{2\rho}\,\tilde{Q}^n{}_{mb}(\tilde{\Gamma})g_{\beta\delta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}u^\beta v^m\left(\frac{q^b}{m_0}\right)\\
&\quad-\bar{Q}^n{}_{\mu\zeta}(\bar{\Gamma})v^\mu u^\zeta\left(-\frac{1}{\rho^3}\,\partial_\nu\rho\left(\frac{q^a}{m_0}\right)+L^a{}_{\beta\nu}u^\beta+\frac{1}{2\rho^3}\,g_{\delta\nu}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\left(\frac{q^a}{m_0}\right)\right)\\
&\quad-\frac{1}{\rho}g_{\zeta\mu}\tilde{g}^{(\zeta\omega)}\rho_{,\omega}\left(\tilde{\nabla}_b\left(\frac{q^a}{2\rho^2 m_0}\right)+\frac{1}{\rho}\,g_{\beta\delta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}u^\beta\delta^a{}_b\right)\left(v^\mu\left(\frac{q^b}{2\rho^2 m_0}\right)-v^b u^\mu\right)\\
&\quad-2(H^n{}_{\mu\nu}-L^n{}_{\mu\nu})\left(\tilde{\nabla}_n\left(\frac{q^a}{2\rho^2 m_0}\right)+\frac{1}{\rho}\,g_{\beta\delta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}u^\beta\delta^a{}_n\right)v^\mu u^\nu
\end{aligned}
$$

$$-2\mu\rho\tilde{g}^{(\nu\beta)}\rho_{,\beta}k_{bc}\left(-\frac{1}{\rho^3}\left(\frac{q^a}{m_0}\right)\rho_{,\nu}+L^a{}_{\gamma\nu}u^\gamma+\frac{1}{2\rho^3}g_{\delta\nu}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\left(\frac{q^a}{m_0}\right)\right)$$
$$\left(\frac{v^b q^c}{2\rho^2 m_0}\right)$$
$$-(2\rho^2 l_{bd}g^{\nu\beta}H^d{}_{\gamma\beta}+\rho^2(l_{bd}g^{\nu\beta}+l_{db}g^{\beta\nu})L^d{}_{\beta\gamma})$$
$$\times\left(-\frac{1}{\rho^3}\left(\frac{q^a}{m_0}\right)\rho_{,\nu}+L^a{}_{\gamma\nu}u^\gamma+\frac{1}{2\rho^3}g_{\delta\nu}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}\left(\frac{q^a}{m_0}\right)\right)\left(v^\gamma\left(\frac{q^b}{2\rho^2 m_0}\right)-v^b u^\gamma\right)$$
$$+\frac{1}{2\rho^3}g_{\beta\delta}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}v^\beta\left(\frac{q^b}{m_0}\right)+L^a{}_{\beta\nu}v^\beta u^\nu+\frac{1}{\rho}g_{\delta\nu}\tilde{g}^{(\delta\gamma)}\rho_{,\gamma}v^a u^\nu=0 \tag{14.57}$$

Simultaneously, equation (14.1) is satisfied and

$$q^b=\text{const} \tag{$*$}$$

is an integral of motion. Moreover, in this case we consider the flow of geodesic $\Gamma(\sigma)$, $\sigma\in U\subset R'$, given by

$$x^A=x^A(\tau,\sigma)$$

and

$$\zeta^A=\left(\frac{\partial x^\alpha}{\partial\sigma},\frac{\partial x^\alpha}{\partial\sigma}\right)\Bigg|_{\sigma=\sigma_0},\qquad \sigma_0\in U$$

σ is a parameter such that for every $\sigma_1\neq\sigma_2$, $x^A(\tau,\sigma_1)$ and $x^A(\tau,\sigma_2)$ are different geodesics. One can say that we have a family of geodesic curves, $\Gamma(\sigma)$. The geodesic considered here is $\Gamma(\sigma_0)$, i.e., for $\sigma=\sigma_0$. Thus,

$$\frac{dx^a}{d\tau}=u^a=\frac{1}{2\rho^2}\left(\frac{q^a}{m_0}\right) \tag{14.58}$$

where $\rho=\rho(\tau,\sigma)=\rho(x(\tau,\sigma))$. Thus, one gets

$$x^a=\frac{1}{2}\left(\frac{q^a}{m_0}\right)(\sigma)\int_{\tau_0}^{\tau}\frac{d\tau}{\rho^2(\tau,\sigma)}+x_0^a(\sigma) \tag{14.59}$$

Thus,

$$\zeta^a=\frac{\partial}{\partial\sigma}\left(\frac{q^a}{m_0}(\sigma)\int_{\tau_0}^{\tau}\frac{d\tau}{\rho^2(\tau,\sigma)}\right)\Bigg|_{\sigma=\sigma_0}+\frac{dx_0^2}{d\sigma} \tag{14.60}$$

$$v^a=\frac{d\zeta^a}{d\tau}=\frac{\partial}{\partial\sigma}\left(\frac{q^a}{m_0}(\sigma)\frac{1}{\rho^2(\tau,\sigma)}\right)\Bigg|_{\sigma=\sigma_0} \tag{14.61}$$

v^a is (of course) an Ad-type quantity. In this way we get

$$\frac{dv^a}{d\tau}=-\frac{\partial}{\partial\sigma}\left(\frac{q^a}{m_0}(\sigma)\cdot\frac{1}{\rho^3}\cdot\frac{\partial\rho}{\partial\tau}\right)\Bigg|_{\sigma=\sigma_0} \tag{14.62}$$

In this way equation (14.56) together with equation (14.1) gives us an interpretation of geodesic deviation equations in N^2AJT^2. They are analogous to the deviation equation for charged particles moving in a non-Abelian Yang-Mills field and nonsymmetric gravitational field as well.

Let us remark on a physical interpretation of the vector $\zeta^A = (\zeta^\alpha, \zeta^a)$. The vector ζ^A, "geodesic separation," is the displacement (tangent vector) from a point on the fiducial geodesic to a point on a nearby geodesic characterized by the same value of the affine parameter τ. Thus, $v^A = (v^\alpha, v^\alpha)$ means a relative "velocity" and $u^B \nabla_B u^A$ a relative "acceleration" equal, according to equation (14.53), to a commutator of covariant derivatives. Thus, we get "tidal forces" in N^2AJT^2 [$(n+4)$-dimensional case], i.e., for charged (in the non-Abelian gauge field sense) test particles in N^2AJT^2. In this equation we get gravitational "field forces" from NGT, Yang-Mills "tidal forces," and additional effects which can be treated as gravito-Yang-Mills tidal forces. The scalar field ρ is also a source of additional "tidal forces."

These new effects are "interference effects" between gravitational and non-Abelian Yang-Mills interactions described by N^2AJT^2. The commutator in (14.53) can be treated as the $(n+4)$-dimensional analogue of "tide producing gravito-Yang-Mills forces." We can try to project our equations on a space-time E (they are defined on a bundle manifold P), taking any local section e of the bundle P. In this way we get gauge-dependent charges Q^a and gauge-dependent $\bar{v}^a$.

We can substitute $\bar{v}^a$ into equations (14.56)-(14.57). However, we should substitute in place of $dv^a/d\tau$ the expression

$$\frac{d\bar{v}^a}{d\tau} - C^a{}_{bc} A^c{}_\mu u^\mu v^b \tag{14.63}$$

where as usual $e^*\omega = A^a{}_\mu \bar{\theta}^\mu X_a$.

Finally, we remark that equation (14.56) represents tidal gravito-Yang-Mills forces and equation (14.57) is the relative change of (q^a/m_0) (σ) for different test particles via v^a [or $(Q^a/m_0)(\sigma)$ via $\bar{v}^a$].

15. CONCLUSIONS AND PROSPECTS

Thus, we get a theory which unifies gravity, gauge fields, and scalar forces. The gravitational field in this theory is described by a nonsymmetric, real tensor $g_{\mu\nu}$ (and a scalar field Ψ), which connects it with Moffat's theory of gravitation (one of the most important alternative theories of gravitation; see ref. 34 for a review). The nonsymmetric Kaluza-Klein (Jordan-Thiry) theory has been previously designed as a unification of Moffat's theory of gravitation and the electromagnetic (or Yang–Mills) field. However, there are some differences. First of all, Moffat and his co-workers use extensively the Einstein–Strauss theory (see ref. 35), but not the Einstein–Kaufman theory. The Einstein–Strauss theory in its hypercomplex version cannot be extended in any simple way to higher dimensions, even in the five-dimensional (electromagnetic) case. It is also a hard task to incorporate spin sources in the Einstein–Strauss theory. In both cases, we meet a fundamental physical problem. The Lagrangian becomes hypercomplex (not real). In our case we do not have these problems because everything is real. In the case of the nonsymmetric Jordan–Thiry theory, we effectively get the scalar–tensor theory of gravitation in the nonsymmetric version. The scalar field behaves very well in the linear approximation. It has been proved (see ref. 29) that we could avoid tachyons and ghosts in the particle spectrum of the theory (if we put $m > 0$). In the case of classical Jordan–Thiry theory, the scalar field is a ghost (a negative kinetic energy). This new version of the Kaluza–Klein theory is capable of removing singularities from the solution of coupled gravitational and Yang–Mills equations even in the case of spherical symmetry. Such solutions have been found in the electromagnetic case (see refs. 30 and 31). It is well known that in the case of the Einstein–Maxwell equations we cannot get any nonsingular, localizable, stationary solutions (the so-called Hilbert-Levi-Civita-Thiry-Einstein-Lichnerowicz-Pauli theorem; see refs. 36–39). This result has been recently extended to the case of non-Abelian gauge fields (see ref. 40). Recently some particlelike solutions of Einstein–Yang–Mills equations have been found (see Bartnik and McKinnon, ref. 40). However, they are magnetic monopole-like solutions and not of electric type.

Quiet recently, Mann (ref. 31) found eight classes of spherically symmetric and stationary solutions in the nonsymmetric Kaluza–Klein theory. These solutions are more general than those from ref. 30 and some of them have no singularities in gravitational and electromagnetic fields. Some of these solutions possess a nonzero magnetic field and nonzero $g_{[29]} = f \neq 0$. The nonsingular solutions are parametrized by fermion charge l^2, electric charge Q, and a new constant u_0. This constant is related to $g_{[23]}$ in a similar way that l^2 is to $g_{[14]}$. It plays a similar role for $g_{[\mu\nu]}$ as magnetic charge for $F_{\mu\nu}$. We recall that the first exact solution found in ref. 30 has no singularity in an electric field and a finite energy. However, it has a weak singularity in $g_{[\alpha\beta]}$. In this case we put $g_{[23]} = 0$. It seems that we can extend these solutions without any problems to the non-Abelian case.

Thus, we can look for models of elementary particles as solutions of field equations.

In the theory there are two field strengths for the electromagnetic (Yang-Mills) field, $F_{\mu\nu}$, $H_{\mu\nu}$ $(H^a{}_{\mu\nu}, L^a{}_{\mu\nu})$. The first is built from $(\mathbf{E}, \mathbf{B})$

$[(\mathbf{E}^a, \mathbf{B}^a)]$, the second from $(\mathbf{D}, \mathbf{H})$ $[(\mathbf{D}^a, \mathbf{H}^a)]$. The relations between both tensors are given by equation (6.2).

According to current ideas (see refs. 103–105) the confinement of color could be connected to the dielectricity of the vacuum (dielectric model of confinement). Due to the so-called antiscreening mechanism, the effective dielectric constant is equal to zero. This means that the energy of an isolated charge goes to infinity. Now, there are also the so-called classical dielectric models of confinement (ref. 106). The confinement is induced by a special kind of dielectricity of the vacuum such that $\mathbf{E} \neq 0$ and $\mathbf{D} = 0$ $(\mathbf{E}^a \neq 0, \mathbf{D}^a = 0)$. In this case we do not have a distribution of charge. This is similar to the electric-type Meissner effect.

It is easy to see that in our case (the nonsymmetric Kaluza–Klein theory) the dielectricity is induced by the nonsymmetric tensors $g_{\mu\nu}$ and l_{ab}. If $g_{[\mu\nu]} = 0$, $\mathbf{D} = \mathbf{E}$ and $\mathbf{B} = \mathbf{H}$.

The gravitational field described by the nonsymmetric tensor $g_{\mu\nu}$ behaves as a medium for an electromagnetic field (gauge field). The conditions $\mathbf{E} \neq 0$, $\mathbf{D} = 0$ $(\mathbf{E}^a \neq 0, \mathbf{D}^a = 0)$ can be satisfied in the axial, stationary case for $F_{\mu\nu}$, $H_{\mu\nu}$ $(H^a{}_{\mu\nu}, L^a{}_{\mu\nu})$, $g_{\mu\nu}$. Thus, it is interesting to find the exact solution with axial symmetry for the nonsymmetric Kaluza–Klein theory with fermion sources for $G = SU(3)_c$. This could offer us a model of hadrons.

The axially symmetric, stationary case seems to be very interesting from a more general point of view. We have in general relativity very peculiar properties of stationary, axially symmetric solutions of the Einstein–Maxwell equations. These solutions describe the gravitational and electromagnetic fields of a rotating charged mass. Thus, we get the magnetic field component. Asymptotically (these solutions are asymptotically flat) the magnetic field behaves as a dipole field. We can calculate the gyromagnetic ratio at infinity, i.e., the ratio of the magnetic dipole moment and the angular momentum moment. It is worth noticing that we get the anomalous gyromagnetic ratio (see Kramer *et al.*, ref. 107), i.e., the gyromagnetic ratio for an electron (for a charged Dirac particle). We cannot interpret the Kerr–Newman solution as a model of the fermion, for we have a singularity. In the nonsymmetric Kaluza–Klein theory we can expect completely nonsingular solutions (refs. 30 and 31). We can also expect the asymptotic behavior of the Einstein–Maxwell theory. Thus, it seems that we probably get solutions with an anomalous gyromagnetic ratio. Such a solution could be treated as a model (classical) a spin-$\frac{1}{2}$ particle. In the non-Abelian case $[G = SU(3)_c \times U(1)_{\mathrm{em}}]$ this could offer us a model of a charged baryon (i.e., proton), where the skewon field $g_{[\mu\nu]}$ induces a confinement of color. In this way, the skewon field $g_{[\mu\nu]}$ plays a double role: (1) additional gravitational interactions (from Moffat's theory of gravitation), (2) a strong interaction field connected to the confinement problem.

It has been proved by Mann and Moffat (see refs. 108, 109) that the skewon field $g_{[\mu\nu]}$ has zero spin. In a linear approximation it is the so-called generalized Maxwell field (an Abelian gauge field). Thus, it is natural to expect an exchange of some spin-zero particles in the nucleon-nucleon potential for low and intermediate energies. We do not observe such particles. However, we cannot fit experimental data for the nucleon-nucleon interaction without the mysterious σ (spin-zero) particles (see, for example, refs. 110, 111).

It happens that we need two such particles to fit the data. In our proposal, they are connected to the skewon field $g_{[\mu\nu]}$ and to the scalar field Ψ from the nonsymmetric Jordan-Thiry theory. The reason we do not detect such particles directly seems to be clear now. They are confined, because they are actually a cause of confinement. The scalar field from the nonsymmetric Jordan-Thiry theory induces an additional dielectricity of the vacuum [see Lagrangians for the scalar field Ψ and for the Yang-Mills' field in equations (8.1), (8.7), (10.6)]. Note that a function of the scalar field Ψ appears as a factor before the Yang-Mills Lagrangian in equation (8.7). This has some important consequences: the effective gravitational "constant" depends on Ψ and in the flat space limit $g_{\mu\nu}=\eta_{\mu\nu}$ the Lagrangian resembles the bosonic part of the soliton bag model Lagrangian if we put

$$e^{i10\Psi}=2\left(1-\frac{\sigma}{\sigma_0}\right);\qquad \sigma_0=\text{const} \tag{15.1}$$

for $n=8$, $G=SU(3)$ [see refs. 112, 113).

One finds

$$\Psi=-\frac{1}{10}\ln\left(\frac{1-\sigma}{\sigma_0}\right)-\frac{\ln 2}{10} \tag{15.2}$$

and in the flat space limit one easily gets

$$\begin{aligned}\mathscr{L}=&-\frac{1}{4}\left(1-\frac{\sigma}{\sigma_0}\right)(h_{ab}+\mu^2k^c{}_bk_{ca})H^a{}_{\mu\nu}H^{b\mu\nu}\\&+\frac{\sigma_0\rho(\mu)}{16\pi(\sigma_0-\sigma)}+\frac{m\sigma_0^2}{100(\sigma_0-\sigma)^4}\,\eta^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu}\end{aligned} \tag{15.3}$$

The full Lagrangian (8.7) is more general and it contains a gravitational field.

Friedberg and Lee (see ref. 114) consider the soliton bag model with a more general factor $K(\sigma)$,

$$\mathscr{L}=-\tfrac{1}{4}K(\sigma)h_{ab}H^{a\mu\nu}H^b{}_{\mu\nu}-\tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma-U(\sigma) \tag{15.4}$$

They consider that the scalar field σ is a new dynamical field with self-interaction given by $U(\sigma)$. The quantity K is a dielectric constant which depends on σ. It is interesting to observe many similarities between (15.4) and our Lagrangian from the nonsymmetric Jordan–Thiry theory, i.e., (8.7). Thus, in our model we have in the flat space limit an effective dielectric constant

$$K_{\text{eff}} = 4\, e^{-10\Psi} \tag{15.5}$$

It is interesting to notice that the scalar field Ψ enters into the effective gravitational "constant" and into the effective dielectric "constant" in the flat space limit.

We recall that in a full nonsymmetric Jordan–Thiry theory (curved non-Riemannian space-time) we have the following symmetry for the scalar field (see refs. 19, 24):

$$\Psi \to \Psi' = f(\Psi) \tag{15.6}$$

where f is an arbitrary function. In this way the formulas (15.1) and (15.6) can be treated as transformations for a scalar field in the nonsymmetric Jordan–Thiry theory. Thus, we can connect a bosonic part of some soliton bag model Lagrangians via equation (15.6) in the nonsymmetric Jordan–Thiry theory. In this way we see some possibilities of connecting gravitational and strong interactions via the nonsymmetric Kaluza–Klein (Jordan–Thiry) theory. This is a little in the spirit of the idea of strong gravity (see ref. 115). It is easy to see that in the nonsymmetric Kaluza–Klein (Jordan–Thiry) theory there are two metric tensors $g_{(\alpha\beta)}$ and $f_{\alpha\beta}$ such that

$$f_{\alpha\beta} g^{(\alpha\gamma)} = \delta^{\gamma}{}_{\beta}; \qquad g_{\alpha\beta} g^{\alpha\gamma} = g_{\beta\alpha} g^{\gamma\alpha} = \delta^{\gamma}{}_{\beta} \tag{15.7}$$

and it is easy to see that if $g_{[\alpha\beta]} = 0$, then $f_{\alpha\beta} = g_{(\alpha\beta)}$.

Thus, we propose the Lagrangian of the nonsymmetric Jordan–Thiry theory as the bosonic part of the Lagrangian of strong interactions. Why? It seems that something is missing in the QCD Lagrangian. We have the following objectives:

1. σ particles (which we mentioned before).
2. An exact solution with color radiation (this means color at infinity—no confinement) found by Tafel and Trautman (see ref. 116).

Thus, it seems that the QCD Lagrangian is incomplete in the bosonic part. In our proposal, we replace the QCD Lagrangian by the Lagrangian from the nonsymmetric non-Abelian Jordan–Thiry theory for $G = SU(3)_c$. In this way we can get a dielectric model of confinement and a soliton bag model-like Lagrangian (see refs. 112–114 and 117).

Thus, we propose the following program of investigation:

1. Find exact solutions of the nonsymmetric Kaluza–Klein and Jordan–Thiry theory in Abelian and non-Abelian cases with and without fermion

sources in the case of spherical and axial symmetry, using inverse scattering, and the Riemann invariants method (ref. 118), and examine their classical stability (for example, Poincaré stability).

2. To find an effective interaction of two axially symmetric solutions exactly, or, using some numerical methods in the case of $G = SU(3)_c$, with fermion sources. This could be similar to the nucleon-nucleon interaction in the Skyrme model. The solutions should be treated as particles using a collective coordinate method.

3. To find wavelike solutions of the field equations in the Abelian and non-Abelian cases. This could, in the electromagnetic case, offer a solution which could be treated as a kind of electromagneto-gravitational wave (nonlinear wave) with nontrivial interactions between all fields. The objective of this hope is related to points 4 and 5 in the list of "interference effects" (we recall that the displacement current in the classical Maxwell equations leads to the nontrivial interaction between the electric and magnetic fields—the *raison d'être* of the wave solutions for the Maxwell equations; however, this is only a historical remark). By a nontrivial interaction, we mean that the flow of energy is possible from one field to another in a quasiperiodic way.

One can try to use the following Ansatz for the simplest gravito-electromagnetic wave in our theory (see ref. 107)

$$\bar{g} = g_{(\alpha\beta)}\, dx^\alpha \otimes dx^\beta = ds^2 = Q(dx^2 + dy^2) - 2\, du \otimes dv - 2H\, du^2$$

$$g = A\, du \wedge dx + B\, du \wedge dy = g_{[\mu\nu]}\, dx^\mu \wedge dx^\nu$$

$$F = C\, du \wedge dx + D\, du \wedge dy = \tfrac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

$$H = E\, du \wedge dx + K\, du \wedge dy = \tfrac{1}{2} H_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

where $u = z - t$, $v = z + t$, and

$$H = H(x, y, u) \qquad Q = Q(x, y)$$

$$A = A(x, y, u) \qquad B = B(x, y, u)$$

$$C = C(x, y, u) \qquad D = D(x, y, u)$$

$$E = E(x, y, u) \qquad K = K(x, y, u)$$

are arbitrary functions of their variables. In this case we expect that $H_{\mu\nu} \neq F_{\mu\nu}$ and the polarization tensor $M_{\mu\nu}$ is not zero.

There are also some proposals concerning cosmology:

To find a cosmological solution of Bianchi type I in the nonsymmetric Kaluza-Klein theory with material sources (ref. 25). Homogeneous, plane symmetric, i.e., Bianchi type I space-time in comoving coordinates, has the metric

$$g_{\mu\nu} = \begin{pmatrix} -\alpha(t) & 0 & 0 & w(t) \\ 0 & -\beta(t) & 0 & 0 \\ 0 & 0 & -\beta(t) & 0 \\ -w(t) & 0 & 0 & 1 \end{pmatrix} \tag{15.8}$$

The electromagnetic strength tensor $F_{\mu\nu}$ has only two nonzero components,

$$F_{14} = E(t), \qquad F_{23} = B(t)$$

The same is true for $H_{\mu\nu}$,

$$H_{14} = D(t), \qquad H_{23} = H(t)$$

One easily gets $E(t) = D(t)$ and $B(t) = H(t)$. The Bianchi identity yields $B = B_0 = \text{const}$.

Thus, the cosmological model in the nonsymmetric Kaluza–Klein theory is described by $\alpha(t)$, $\beta(t)$, $w(t)$, $E(t)$, and a constant B_0. For a perfect fluid cosmology we should take

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} \tag{15.9}$$

where the velocity four-vector u^μ is in comoving coordinates given by

$$u^i = 0$$

$$u^4 = 1$$

$$g_{\mu\nu}u^\mu u^\nu = 1$$

$$T = g^{\mu\nu}T_{\mu\nu} = \rho - 3p$$

The four generalized Bianchi identities on $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ give rise to the set of covariant conservation laws

$$((-g)^{1/2}g^{\alpha\nu}T_{\rho\nu} + (-g)^{1/2}g^{\nu\alpha}T_{\nu\rho})_{,\alpha} + g^{\mu\nu}{}_{,\rho}(-g)^{1/2}T_{\mu\nu} = 0 \tag{15.10}$$

We expect a completely nonsingular solution of the field equations and equation (15.10) for (15.9) and (15.8). It seems that the nonsingular behavior will be better than for nonsingular solutions in the Einstein–Cartan theory, where the metric and density of matter are nonsingular, but torsion and spin are singular.

We propose a program of research which consists in finding exact solutions in this theory. These solutions could be treated as models of particles (generalized skyrmions; ref. 119). Our approach seems to be more realistic, because we include into the Lagrangian both gauge and gravitational fields. In the Skyrme model we have to deal with an effective model of strong interactions. This model, despite many spectacular successes, has some problems. For example, a mass difference between nucleon and Δ^{2+} particle. Moreover, the interactions between two skyrmions can give a qualitatively good description of a nucleon–nucleon potential (ref. 111). In this way we could approach some classical nuclear phenomenology as in ref. 120. Moreover, there is a problem with a central attractive potential in the model. Our approach probably could improve this fact.

One could search for axially symmetric, stationary solutions in the nonsymmetric Kaluza–Klein (Jordan–Thiry) theory using the formalism presented in ref. 121. Looking for axially symmetric, stationary solutions in the nonsymmetric Kaluza–Klein theory, we can try the following form for $g_{[\mu\nu]}$, $F_{\mu\nu}$, $H_{\mu\nu}$, and $g_{(\mu\nu)}$:

$$g_{[\mu\nu]}=\begin{pmatrix} 0 & 0 & a\alpha e^{v} & \alpha e^{v} \\ 0 & 0 & a\beta e^{v} & \beta e^{v} \\ -a\alpha e^{v} & -a\beta e^{v} & 0 & 0 \\ -a\alpha e^{v} & -a\beta e^{v} & 0 & 0 \end{pmatrix} \tag{15.11}$$

$$F_{\mu\nu}=\begin{pmatrix} 0 & 0 & -B_z & -E_\rho \\ 0 & 0 & B_\rho & -E_z \\ B_z & -B_\rho & 0 & 0 \\ E_\rho & E_z & 0 & 0 \end{pmatrix} \tag{15.12}$$

$$H_{\mu\nu}=\begin{pmatrix} 0 & 0 & -H_z & -D_\rho \\ 0 & 0 & H_\rho & -D_z \\ H_z & -H_\rho & 0 & 0 \\ D_\rho & D_z & 0 & 0 \end{pmatrix} \tag{15.13}$$

and

$$g_{(\alpha\beta)}\, dx^\alpha \otimes dx^\beta = ds^2 = e^{\mu}(dz^2+d\rho^2) + X\, d\rho^2 + 2W\, d\varphi \otimes dt - V\, dt^2 \tag{15.14}$$

where all the functions a, α, β, u, v, B_z, B_ρ, E_ρ, E_z, H_ρ, H_z, D_ρ, D_z, X, V, and W depend on ρ and z only. In this case we expect that $F_{\mu\nu} \neq H_{\mu\nu}$ and we will get a nonzero polarization tensor $M_{\mu\nu}$.

Finally, let us reconsider equation (10.14) and rewrite it in a more convenient way using $M^a{}_{\alpha\beta}$ defined in Section 10,

$$T_c(g) = l_{dc} g_{\delta\beta} g^{\gamma\delta} M^d{}_{\gamma\alpha} + l_{cd} g_{\alpha\delta} g^{\delta\gamma} M^d{}_{\beta\gamma} - \frac{1}{4\pi}\left(l_{cd} g_{\delta\beta} g^{\gamma\delta} H^d{}_{\gamma\alpha} - l_{dc} g_{\alpha\beta} g^{\delta\gamma} H^d{}_{\beta\gamma}\right)$$
$$= 0 \tag{15.15}$$

Equation (15.15) can be rewritten in a matrix form:

$$T(g)=g(g^{-1})^T\cdot(l*\mathbf{M})+g^Tg^{-1}(l^T*\mathbf{M}^T)$$
$$-\frac{1}{4\pi}(g(g^{-1})^T(l*\mathbf{H})-g^Tg^{-1}(l^T*\mathbf{H}^T))$$
$$=0 \tag{15.15a}$$

where "T" means a matrix transposition and "$*$" the action of an $n\times n$ matrix on an n-dimensional vector.

According to equation (10.14), the tensor $L^a{}_{\alpha\beta}$ is expressible by $H^a{}_{\alpha\beta}$ and $g_{\mu\nu}$. The equation is linear with respect to $L^a{}_{\alpha\beta}$ and can be solved. The quantity $M^a{}_{\alpha\beta}$ has the physical interpretation as the polarization tensor for the Yang–Mills field. Simultaneously, we get the geometrical interpretation of $M^a{}_{\alpha\beta}$ as a torsion in higher dimensions ($Q^a{}_{\alpha\beta}=8\pi M^a{}_{\alpha\beta}$). Thus, we come to the conclusion that it would be possible to reinterpret the theory as a theory with nonzero torsion in higher dimensions as a fundamental quantity. In this way the tensor $g_{\alpha\beta}$ is a solution of equation (15.15) and $H^a{}_{\mu\nu}$ and $M^a{}_{\mu\nu}$ are known quantities. Moreover, equation (15.15) is nonlinear with respect to $g_{\mu\nu}$ and, because of this, more difficult to solve. In this way we reinterpret the full theory as a theory with torsion in higher dimensions. Thus, our theory has many similarities with previous approaches, i.e., the Kaluza–Klein theory with torsion (see refs. 16 and 17).

Equation (15.15) can be considered a system of nonlinear equations for $g=(g_{\alpha\beta})\in X$. Moreover, we have to deal with n transformations defined in $D(T_\alpha)\subset X$, i.e., $T_a\colon X\to X$, $a=1,2,\ldots,n$. Equation (15.15) says that

$$\hat{g}\in D(T_a)=D(T)=\{g,\det(g_{\alpha\beta})\neq 0\}$$

is a common root of n transformations T^a. If such a system of equations is coherent, we can try to solve the system and to find $\hat{g}$. The system is coherent in an open neighborhood of $h_0\in D(T)$ if

$$\operatorname{Rank}\begin{vmatrix} dT_{1|g=h_0}\\ dT_{2|g=h_0}\\ \vdots\\ dT_{n|g=h_0}\end{vmatrix}=16 \tag{15.16}$$

$dT_{a|g=h_0}\in B(\mathscr{X},\mathscr{X})$, and

$$\operatorname{Rank}|dT_{a|g=h_0}|=16 \tag{15.17}$$

for every $a=1,2,\ldots,n$.

Thus, if equations (15.16) and (15.17) are satisfied, we can try to find $\hat{g}$. Let us notice that if $\hat{g}$ satisfies equation (15.15), then $\varkappa\hat{g}$, $\varkappa\neq 0$, satisfies it as well. Equation (15.15) can be solved using iterative methods. The convenient method seems to be the generalized Newton method, i.e., the method of W. L. Kantorowitch, as in the electromagnetic case (five-dimensional).

If the conditions (15.16) and (15.17) are satisfied, then one can choose any $1 \le a_0 \le n$ and consider the transformation $T_{a0}\colon X \to X$. The solution of equation (15.15) can be obtained from $T_{a0}(\hat{g}) = 0$ using the Kantorowitch method. In particular we construct a sequence

$$\overset{(b)}{g} = h_0$$

$$\overset{(n+1)}{g} = \overset{(h)}{g} - [(dT_{a0})|_{\overset{(n)}{g}}]^{-1} T^{a0}(\overset{(n)}{g}) \tag{15.18}$$

or

$$\overset{(0)}{g'} = h_0$$

$$\overset{(n+1)}{g'} = \overset{(h)}{g'} - [(dT_{a0})|_{h_0}]^{-1} T^{a0}(\overset{(n)}{g'}) \tag{15.18'}$$

looking for a limit

$$\overset{(\infty)}{g} = \lim_{n=\infty} \overset{(n)}{g}$$

or

$$\overset{(\infty')}{g} = \lim_{n=\infty} \overset{(n)}{g'}$$

If the sequences converge, one has

$$T_{a0}(\overset{(\infty)}{g}) = T_{a0}(\overset{(\infty)}{g'})$$

and of course, because of equations (15.16)-(15.17),

$$T_a(\overset{(\infty)}{g}) = T_a(\overset{(\infty)}{g'}) = 0 \qquad \text{for} \quad a = 1, 2, \ldots, n$$

The sequence (15.18) converges faster then (15.18').

One easily gets

$$\begin{aligned}((dT_{a0})|_g)^{\mu\nu}{}_{\alpha\beta} &= l_{da_0}(\delta^{\nu}{}_{\beta} g^{\gamma\mu} - g_{\delta\beta} g^{\gamma\nu} g^{\mu\beta}) \left(m^d{}_{\gamma\alpha} - \frac{1}{4\pi} H^d{}_{\gamma\alpha} \right) \\ &\quad + l_{da_0}(\delta^{\mu}{}_{\alpha} g^{\nu\gamma} - g_{\alpha\delta} g^{\delta\nu} g^{\mu\gamma}) \left(M^d{}_{\beta\gamma} - \frac{1}{4\pi} H^d{}_{\beta\gamma} \right)\end{aligned} \tag{15.19}$$

and similarly $(d^2T_{a0})_{|g}$, which we do not write here.

Finally, we get

$$h_{\mu\nu} = \overset{(\infty)}{g}_{[\mu\nu]}(M^d{}_{\psi\varphi}, H^d{}_{\rho\alpha})$$

i.e., the skewon field induced by the Yang-Mills field and the polarization tensor.

Let us note that we can proceed in the following way. We can calculate $g_{[\mu\nu]}$ induced by an electromagnetic field $F_{\mu\nu}$ and its polarization $M_{\mu\nu}$. After this we can substitute this tensor in equation (10.14), getting the Yang-Mills polarization tensor. In this way $F_{\mu\nu}$ and $M_{\mu\nu}$ induce the skewon field and the polarization of the Yang-Mills field.

APPENDIX A

Let us consider a more general form of the nonsymmetric metric on P in a lift horizontal basis,

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & P_{ab} \end{array}\right) \tag{A.1}$$

$$\gamma = \pi^* g \oplus \beta(\omega, \omega)$$

where P_{ab} ($\hat{P} = P_{ab}\theta^a \otimes \theta^b$) is a nonsymmetric invertible tensor on a group G (negatively defined, right-invariant) and in general it does depend parametrically on a point on a space-time E. In Section 4 it is supposed that

$$P_{ab} = \rho^2 l_{ab} = \rho^2(h_{ab} + \mu k_{ab}) \quad (P = \rho^2 l) \tag{A.2}$$

where $\rho = \rho(x)$ is a scalar field on E, h_{ab} is a Killing-Cartan tensor on G, and k_{ab} is a skew-symmetric right-invariant form on a group G. Now the general shape of an affine connection $\omega^A{}_B$ defined on $\underline{P}$ compatible in the Einstein-Kaufman sense with the nonsymmetric metric on $\underline{P}$ is

$$\omega^A{}_B = \left(\begin{array}{c|c} \pi^*(\bar{\omega}^\alpha{}_\beta) - P_{dc} g^{\alpha\sigma} L^d_{\alpha\beta}\theta^c & L^\alpha_{\beta\gamma}\theta^\gamma - P^{ae} g_{\beta\delta} N^\delta_{de}\theta^d \\ \hline P_{bd} g^{\sigma\alpha}(2H^d_{\gamma\sigma} - L^d_{\gamma\sigma})\theta^\gamma + N^\alpha_{bc})\theta^c & -P^{ea} g_{\delta\sigma} N^\delta_{eb}\theta^\sigma + \tilde{\omega}^a{}_b \end{array}\right) \tag{A.3}$$

where

$$\tilde{\omega}^a{}_b = \tilde{\Gamma}^a{}_{bc}\theta^c, \qquad P_{ab}P^{cb} = p_{ba}P^{bc} = \delta^c{}_a$$

Compatibility conditions (4.7) for $\omega^A{}_B$ read, in terms of $L^d{}_{\sigma\beta}$, $N^\delta{}_{de}$ (which are Ad-type quantities), and $\tilde{\Gamma}^a{}_{bc}$ (which is a right-invariant connection of G)

$$P_{db}\tilde{\Gamma}^d{}_{ac} + P_{ad}\tilde{\Gamma}^d{}_{cb} = P_{ab,c} - P_{ad}C^d{}_{bc} \tag{A.4}$$

$$g_{\delta\gamma}P_{db}P^{cd}N^\delta{}_{ca} + g_{\gamma\delta}P_{ad}P^{de}N^\delta{}_{be} = -P_{ab,\gamma} \tag{A.5}$$

$$P_{dc}g_{\delta\beta}g^{\sigma\delta}L^d{}_{\sigma\alpha} + P_{cd}g_{\alpha\delta}g^{\delta\sigma}L^d{}_{\beta\sigma} = 2P_{cd}g_{\alpha\delta}g^{\delta\sigma}H^d{}_{\beta\sigma} \tag{A.6}$$

The connection (A.3) is right-invariant with respect to the right-action of the group G on P.

Writing the geodetic equations (5.1) in $\omega^A{}_B$ [see (A.1)] (i.e., for $\Gamma \subset P$) one easily finds for $A = a$ [i.e., for ver($u(t)$)]

$$u^B \nabla_B u^a = 0 \tag{A.7}$$

[∇_u ver(u) = 0, where u is a tangent vector to Γ] or

$$\frac{du^a}{dt} + L^a{}_{\beta\gamma} u^\beta u^\gamma - u^\beta u^c (P^{de} g_{\beta\delta} N^\delta{}_{ce} + P^{ea} g_{\delta\beta} N^\delta{}_{eb}) + \tilde{\Gamma}^a{}_{bc} u^c u^b = 0 \tag{A.8}$$

In the classical Kaluza-Klein theory the geodetic equation possesses the first integral of motion, i.e.,

$$\frac{du^a}{dt} = 0, \qquad \text{ver}(u(t)) = \text{const} \tag{A.9}$$

Let us suppose that (A.8) has first integrals of motion which are linear functions of u^a, i.e.,

$$\frac{dv^a}{dt} = 0, \qquad v^a = \varkappa^a{}_b u^b, \qquad v = \hat{\varkappa}(\text{ver}(u(t))) \tag{A.10}$$

and

$$u^b = \bar{\varkappa}^b{}_c v^c, \qquad \text{ver}(u(t)) = \hat{\varkappa}^{-1}(v) \tag{A.11}$$

such that

$$\bar{\varkappa}^a{}_b \varkappa^b{}_c = \delta^a{}_c \tag{A.12}$$

and it is bi-invariant [$R^*(g)\hat{\varkappa} = L^*(g)\hat{\varkappa} = \hat{\varkappa}$]. One finds, using (A.10)-(A.12) and (A.8),

$$\begin{aligned}(\varkappa^d{}_{b,c} - \tilde{\Gamma}^e{}_{bc}\varkappa^d{}_e) u^b u^c + u^\beta u^c \varkappa^d{}_a (P^{ae} g_{\beta\delta} N^\delta{}_{ce} \\ + P^{ea} g_{\delta\beta} N^\delta{}_{eb} + \bar{\varkappa}^e{}_f \varkappa^f{}_{e,\beta}) - \varkappa^d{}_a L^a{}_{\beta\gamma} u^\beta u^\gamma = 0.\end{aligned} \tag{A.13}$$

Thus we get

$$L^a{}_{\beta\gamma} = -L^a{}_{\gamma\beta} \tag{A.14}$$

$$(\varkappa^d{}_{(b,c)} + \tilde{\Gamma}^e{}_{(bc)} \varkappa^d{}_e) = 0 \tag{A.15}$$

$$P^{ae} g_{\beta\delta} N^\delta{}_{ce} + P^{ea} g_{\delta\beta} N^\delta{}_{ec} + \bar{\varkappa}^a{}_f \varkappa^f{}_{c,\beta} = 0 \tag{A.16}$$

For $\varkappa^d{}_b$ bi-invariant, it is constant on every fiber and it can depend only on a space-time point. This yields

$$\varkappa^d{}_b = \varkappa(x)\delta^d{}_b, \qquad \varkappa(x) \neq 0 \tag{A.17}$$

$$\bar{\varkappa}^a{}_f = \frac{1}{\varkappa}\delta^a{}_f$$

or $\hat{\varkappa} = \varkappa(x)\,\mathrm{id}_{\mathfrak{g}}$, $\hat{\varkappa}^{-1} = [1/\varkappa(x)]\,\mathrm{id}_{\mathfrak{g}}$, where $\mathrm{id}_{\mathfrak{g}}$ is an identical transformation in the Lie algebra $\mathfrak{g}$ (of G) and (A.16) reads

$$P^{ae} g_{\beta\delta} N^{\delta}{}_{ce} + P^{ea} g_{\delta\beta} N^{\delta}{}_{ec} + \delta^{a}{}_{c} \frac{\varkappa_{,\beta}}{\varkappa} = 0 \tag{A.18}$$

Comparing (A.5) and (A.18) for every $\varkappa$, $g_{\alpha\beta}$, N^{β}_{ab}, and P_{ab}, one gets the following conditions:

$$P_{ab} = \rho^2 l_{ab} \tag{A.19}$$

where l_{ab} does not depend on a space-time point,

$$N^{\delta}{}_{ab} = N^{\delta} l_{ab} \tag{A.20}$$

$$\varkappa = \rho^2 \tag{A.21}$$

where $\rho = \rho(x)$ is a scalar field on a space-time and N^{δ} is a function of the space-time point only. This can be achieved in the following way. Let us consider (A.18) and transform it into

$$g_{\beta\delta}(P^{ae} N^{\delta}{}_{ce} - \tfrac{1}{2} V^{\delta} \delta^{a}{}_{c}) + g_{\delta\beta}(P^{ea} N^{\delta}{}_{ec} - \tfrac{1}{2} V^{\delta} \delta^{a}{}_{c}) = 0 \tag{A.18*}$$

where $V^{\delta} = -(1/\varkappa)\tilde{g}^{(\delta\gamma)}\varkappa_{,\gamma}$, and V^{δ} is a function of x only.

Thus, we get

$$p^{ae} N^{\delta} = \tfrac{1}{2} V^{\delta} \delta^{a}{}_{c}$$
$$p^{ea} N^{\delta}{}_{ec} = \tfrac{1}{2} V^{\delta} \delta^{a}{}_{c} \tag{A.18**}$$

for $g_{\alpha\beta}$ an arbitrary nonsymmetric tensor.

From (A.18**) one easily gets

$$N^{\delta}{}_{ce} = \tfrac{1}{2} V^{\delta} p_{ce} \tag{A.18***}$$

Let us consider equation (A.5), substituting equation (A.18***) into it. One gets

$$(\ln \varkappa)_{,\gamma} \cdot P_{ab} = P_{ab,\gamma} \tag{A.5*}$$

One gets from (A.5*) the following formula:

$$P_{ab} = \varkappa l_{ab}$$

where l_{ab} is right-invariant and does not depend on $x \in E$, i.e., (A.19)–(A.21). Thus, we get the connection (6.1). Demanding the existence of the first integral of motion

$$\gamma((\mathrm{hor}(u(t)), \mathrm{hor}(u(t)))) = g_{(\alpha\beta)}\left(\frac{dx^{\alpha}}{dt}\right)\left(\frac{dx^{\beta}}{dt}\right) = \mathrm{const} \tag{A.22}$$

we get $\rho = \text{const}$ (see Section 4.12). In this way we get Theorems I and II given in the Introduction. Roughly speaking, Theorem I establishes the nonsymmetric (Einstein–Kaufman) G-structure (a right G-structure) with the usual interpretation of the geodetic equations as equations of motion for a test particle, i.e., possessing non-Abelian gauge-independent (and gauge-dependent) charges satisfying the Kerner–Wong–Kopczyński equations. We can repeat all the considerations for a left-invariant structure. If we calculate a curvature scalar density for a connection (A.3), we get the following expression:

$$
\begin{aligned}
|\gamma|^{1/2} R = |\gamma|^{1/2}\{ & \bar{R} + \tilde{R} + [P_{cd} g^{\beta\gamma} g^{\alpha\sigma} L^{c}{}_{\alpha\beta} H^{d}{}_{\sigma\gamma} - 2P_{cd}(g^{[\mu\nu]} H^{c}_{\mu\nu})(g^{[\alpha\beta]} H^{d}_{\alpha\beta})] \\
& + g_{\gamma\delta}(N^{\gamma}{}_{ab} P^{ab})(N^{\delta}{}_{cd} P^{cd}) - P^{fc} P^{ea} g_{\alpha\delta} N^{\alpha}{}_{ca} N^{\delta}{}_{fe} \\
& + P^{fc} P^{ea} g_{[\alpha\delta]} N^{\alpha}{}_{ec} N^{\delta}{}_{fa} + 2P^{af} P^{qc} g_{\gamma\delta} N^{\delta}{}_{fc} N^{\gamma}{}_{qa} \\
& - 2\bar{\nabla}_{\gamma}(N^{\gamma}{}_{ab} P^{ab})\} + \text{full}(n+4) - \text{divergence}
\end{aligned}
\tag{A.23}
$$

Let us consider some different aspects of our theory. Let the tensor $\hat{P}$ defined on the group G be parametrized by $x \in E$, and we do not suppose any invariant properties of the tensor with respect to the action of G. Moreover, we suppose that the connection defined on the fiber bundle of frames over $\underline{P}$ is compatible with the nonsymmetric tensor γ (built with a help of the tensor $\hat{P}$). In this case we have the following theorem:

1. Let conditions 1–4 of Theorem I be satisfied except for the fact that $\hat{P}$ is right-invariant with respect to the right-action of the group G on $\underline{P}$.
2. Let the curvature scalar of the connection $\omega^{A}{}_{B}$, ($\tilde{\omega}$), be invariant with respect to the right-action of the group G on the fiber bundle of frames over ($\underline{P}$, γ) (lifted from the bundle $\underline{P}$).

Let condition 5 from Theorem I be satisfied.

Thus, $\hat{P}$ is right-invariant with respect to the action of the group G and it has a factorization property $\hat{P} = \rho^2 l$, where $\rho = \rho(x)$ is a scalar field on E and $l = l_{ab}\theta^a \otimes \theta^b$ is a right-invariant tensor on the group G.

The proof can be easily obtained directly from the form of the curvature scalar equation (A.23) in the case of symmetric g and arbitrary $H^{a}{}_{\mu\nu}$ (arbitrary ω) modulo equations (A.5), (A.6), (A.14), (A.16), and (A.18).

In Section 3 we mention the right-invariance of the Einstein connection on ($\underline{P}$, γ). What does this mean?

Let $\Phi: G \times \underline{P} \to \underline{P}$ be a right-action of the group G on $\underline{P}$ and let $\Phi^*(g)$ be a contragradient map to $\Phi'(g)$, a tangent map to Φ at $g \in G$. Let $\Sigma: G \to GL(n+4), \mathbb{R})$ be a homomorphism of groups and let us consider a connection $\tilde{\omega}$ on a fiber bundle of frames over $\underline{P}$ with a structural group

$GL(n+4,\mathbb{R})$ compatible in the Einstein–Kaufman sense with the nonsymmetric tensor γ (see Section 1). The connection is right-invariant with respect to the action of the group G on P if one has

$$\hat{\Phi}^*(g)\tilde{\omega} = \mathrm{Ad}_{GL(n+4,\,R)}(\Sigma(g^{-1}))\tilde{\omega}$$

($\hat{\Phi}$ is an action of G on P lifted to this bundle) or for any local section E of the bundle of frames over $\underline{P}$,

$$\Phi^*(g)\Gamma = \mathrm{Ad}_{GL(n+4,\,R)}(\Sigma(g^{-1}))\Gamma + \Sigma^{-1}(g)d\Sigma(g) \tag{A.24}$$

where $\Gamma = E^*\tilde{\omega}$, $\Gamma = \Gamma^A{}_{BC}\theta^C X^B{}_A$, and $X^B{}_A$ are generators of the Lie algebra $gl(R, n+4)$ of the group $GL(n+4,\mathbb{R})$ and $\mathrm{Ad}_{GL(n+4,R)}$ is an adjoint representation of $GL(n+4,\mathbb{R})$. Thus, one gets

$$\begin{aligned}\Phi^*(g)\Gamma^{A'}{}_{B'C'} &= \Sigma^A{}_A(g^{-1})\Gamma^A{}_{BC}\Sigma^B{}_{B'}(g^{-1})\Sigma^C{}_{C'}(g^{-1})\\ &\quad + \Sigma^{-1A'}{}_A(g)d_c\Sigma^A{}_{B'}(g)\Sigma^C{}_{C'}(g)\end{aligned} \tag{A.25}$$

where d_c is a vector field duel to θ^c. The reper transforms

$$\Phi^*(g)\theta^c = \Sigma^C{}_{C'}(g^{-1})\theta^{C'} \tag{A.26}$$

Let us take the lift horizontal basis. In this case one gets

$$\Sigma^A{}_B(g) = \left(\begin{array}{c|c} \delta^\alpha{}_\beta & 0 \\ \hline 0 & U^a{}_b(g) \end{array}\right) \tag{A.27}$$

where $U^a{}_b(g) = (\mathrm{Ad}_G(g))^a{}_b$ is a matrix of the adjoint representation of G. Thus, we have

$$\begin{aligned}\Phi^*(g)\theta^\alpha &= \theta^\alpha\\ \Phi^*(g)\theta^a &= U^a{}_{a'}(g)\theta^{a'}\end{aligned} \tag{A.28}$$

$$\begin{aligned}\Phi^*(g)\Gamma^\alpha{}_{\beta\gamma} &= \Gamma^\alpha{}_{\beta\gamma}\\ \Phi^*(g)\Gamma^{a'}{}_{\beta'\gamma'} &= U^{a'}{}_a(g^{-1})\Gamma^a{}_{\beta'\gamma'}\\ \Phi^*(g)\Gamma^{a'}{}_{\beta'c'} &= U^{a'}{}_a(g^{-1})\Gamma^a{}_{\beta'c}U^c{}_{c'}(g)\\ \Phi^*(g)\Gamma^{a'}{}_{b'\gamma} &= U^{a'}{}_a(g^{-1})\Gamma^a{}_{b\gamma}U^{b'}{}_b(g^{-1})\\ \Phi^*(g)\Gamma^{\alpha'}{}_{\beta'a'} &= \Gamma^a{}_{\beta'a}U^a{}_{a'}(g^{-1})\end{aligned} \tag{A.29}$$

For $\omega^a{}_b = \tilde{\Gamma}^a{}_{bc}\theta^c$ one gets

$$\Phi^*(g)\omega^{a'}{}_{b'} = U^{a'}{}_a(g^{-1})\omega^a{}_b U^b{}_{b'}(g^{-1}) \tag{A.30}$$

Thus, we get the Ad_G property of $L^d{}_{\mu\nu}$, $N^\alpha{}_{bc}$, and equation (7.18) for $\tilde{\Gamma}^a{}_{bc}$. In this way $\tilde{\Gamma}^a{}_{bc}$ has tensorial properties in the lift horizontal basis (Ad type).

Moreover, equation (A.30) has a natural interpretation as a right-invariance of the connection $\tilde{\omega}^a{}_b$ with respect to the right-action of the group G on G. The second equation of (A.28) means the Ad property of the connection on the principal fiber bundle $\underline{P}$ (a gauge bundle).

Equation (A.30) can be rewritten in the more familiar form

$$R^*(g)\tilde{\Gamma} = \mathrm{Ad}_{GL(n,R)}(\tilde{\Sigma}(g^{-1}))\tilde{\Gamma} + \tilde{\Sigma}^{-1}(g)\, d\tilde{\Sigma}(g) \tag{A.31}$$

where $\tilde{\Gamma} = \tilde{\omega}^a{}_b Y^b{}_a$ and $Y^b{}_a$ are generators of the Lie algebra $gl(n, R)$ of the group $GL(n, \mathbb{R})$ and $\mathrm{Ad}_{GL(n,R)}$ is an adjoint representation of $GL(n, \mathbb{R})$ and R is a right-action of G on G. Here

$$\tilde{\Sigma}: \quad G \to GL(n, \mathbb{R}) \tag{A.32}$$

is a smooth homomorphism of groups such that

$$\tilde{\Sigma}^a{}_b(g) = (\mathrm{Ad}_G(g))^a{}_b = U^a{}_b(g) \tag{A.33}$$

In this way we come to the notion of the fiber bundle of frames over a group G and to the right-invariant connection defined on this bundle. Equation (A.31) can be rewritten

$$\hat{R}^*(g)\hat{\omega} = \hat{\omega} \tag{A.34}$$

where $\hat{\omega}$ is a connection on the principal fiber bundle of frames over G with the structural group $GL(n, \mathbb{R})$ and

$$\tilde{\Gamma} = f^*\hat{\omega} \tag{A.35}$$

f is a local section of this bundle. $\hat{R}$ is an action of G on G lifted to the bundle of frames.

Note that our considerations are valid for any connection defined on a fiber bundle of frames over $\underline{P}$, not only for the Einstein–Kaufman one. We can say the same for a connection on a fiber bundle of frames over G. The above considerations justify some Ad_G properties of $\omega^A{}_B$ defined on the manifold $\underline{P}$ (a gauge bundle) and $\tilde{\omega}^a{}_b$ defined on G (a group manifold) (see Section 3). They are treated there as 1-forms defined on P or G according to our conventions from Section 1. From (A.24) one easily gets transformation laws for the curvature

$$\hat{\Phi}^*(g)\tilde{\Omega} = \tilde{\Omega} \tag{A.36}$$

and from (A.34)

$$\hat{R}^*(g)\tilde{\Omega} = \tilde{\Omega} \tag{A.37}$$

For the curvature scalars we get

$$\hat{\Phi}^*(g)R(\tilde{\omega}) = R(\tilde{\omega}) \tag{A.38}$$

$$\hat{R}^*(g)R(\hat{\omega}) = R(\hat{\omega}) \tag{A.39}$$

According to Section 4.5, the connection $\hat{\omega}$ on the fiber bundle of frames over the group manifold G can be induced in the following way. Let us define a principal fiber bundle Π over G with a structural group G and a projection π_G. Let us define a connection $\bar{\Xi}$ on this bundle as right-invariant with respect to the action of the group G on G, i.e.,

$$R^*(g)\bar{\Xi} = \bar{\Xi} \tag{A.40}$$

$$\bar{\Xi} = \bar{\Xi}^a X_a \tag{A.41}$$

Taking any local section f of the bundle, one gets

$$f^*\bar{\Xi} = \Xi^a{}_b v^b X_a \tag{A.42}$$

where V^b are right-invariant forms on G. The quantity $\Xi^a{}_b$ induces in a natural way a connection on the fiber bundle of frames over G [see equations (7.9)–(7.11) for dim $G > 4$ such that

$$\tilde{\Gamma}^e{}_{ab} v^a \wedge v^b = d\Xi^e \tag{A.43}$$

Now we can construct a connection on the bundle Π using the connection defined in Section 4.5. One has

$$\Xi^a{}_b = \frac{1}{2}\,\delta^a{}_b - \frac{\mu}{6}\,k^a{}_b \tag{A.44}$$

in a local section f of Π.

The corresponding curvature can be easily calculated,

$$\bar{\Omega}_G = d\bar{\Xi} + \tfrac{1}{2}[\bar{\Xi}, \bar{\Xi}] \tag{A.45}$$

Thus,

$$f^*\bar{\Omega}_G = d\Xi + \tfrac{1}{2} C^c{}_{ab} \Xi^a{}_d \Xi^b{}_e v^d \wedge v^e \tag{A.46}$$

or

$$\bar{F}^d{}_{ab} = C^d{}_{ef} \Xi^e{}_a \Xi^f{}_b + C^e{}_{ab} \Xi^d{}_e - C^d{}_{eb} \Xi^e{}_a + C^d{}_{ea} \Xi^a{}_b = \tilde{\Gamma}^d{}_{ab} + C^d{}_{ef} \Xi^e{}_a \Xi^f{}_b \tag{A.47}$$

Taking equation (A.44) and substituting it into (A.47), we get

$$\bar{F}^d{}_{ab} = \frac{-1}{4}\,C^d{}_{ab} + \frac{\mu^2}{36}\,C^d{}_{ef} k^e{}_a k^f{}_b - \frac{\mu}{4}\,(C^d{}_{af} k^f{}_b - C^d{}_{eb} k^e{}_a) - \frac{\mu}{6}\,C^p{}_{ab} k^d{}_p \tag{A.48}$$

In the case of $\mu = 0$ (this corresponds to the Riemannian connection on G) one gets

$$\bar{F}^{d}{}_{ab} = -\tfrac{1}{4}C^{d}{}_{ab} \neq 0 \tag{A.49}$$

Changing the section f to a section e, we transform Ξ into $\bar{\Xi}$ in a well-known way (the curvature transforms similarly). Thus, we have it in any local section of Π and we can derive Ξ.

APPENDIX B

Let us consider the equation of motion for a test particle [equation (14.25)] in the limit of the special theory of relativity (i.e., $g_{\alpha\beta} = \eta_{\alpha\beta}$); one gets

$$\frac{du^{\alpha}}{d\tau} - \left(\frac{q^{a}}{m_0}\right)(l_{ab}H^{b\alpha}{}_{\beta}u^{\beta} + \mu k_{ab}L^{b\alpha}{}_{\beta}u^{\beta}) - \frac{\|q\|^2}{8m_0^2}\eta^{\alpha\beta}\left(\frac{1}{\rho^2}\right)_{,\beta} = 0 \tag{B.1}$$

where $q^{a} = \text{const}$ is a gauge-independent charge. Using equation (11.1), one gets

$$\frac{du^{\alpha}}{d\tau} - \left(\frac{q^{a}}{m_0}\right)(h_{ab} - 2\mu k_{ab} - \mu^2 k^{d}{}_{a}k_{db})H^{b\alpha}{}_{\beta}u^{\beta} - \frac{\|q\|^2}{8m_0^2}\eta^{\alpha\beta}\left(\frac{1}{\rho^2}\right)_{,\beta} = 0 \tag{B.2}$$

where $k^{d}{}_{a} = h^{de}k_{ea}$.

This equation has an integral of motion

$$\eta_{\alpha\beta}u^{\alpha}u^{\beta} - \frac{\|q\|^2}{8m_0^2\rho^2} = \text{const} \tag{B.3}$$

Let us find an interpretation of the integral of motion. In order to do this, let us define the four-momentum of a test particle in the usual way:

$$p^{\mu} = m_0 u^{\mu} \tag{B.4}$$

One gets

$$\eta_{\alpha\beta}p^{\alpha}p^{\beta} - \frac{\|q\|^2}{8\rho^2} = \text{const} \tag{B.5}$$

In the case of $\rho = 1$ we should get the usual formula from special relativity, i.e.,

$$\eta_{\alpha\beta}p^{\alpha}p^{\beta} = m_0^2 \tag{B.6}$$

Thus, we obtain

$$\eta_{\alpha\beta}p^{\alpha}p^{\beta} = m_0^2\left[1 + \frac{\|q\|^2}{8}\left(\frac{1}{\rho^2} - 1\right)\right] \tag{B.7}$$

or

$$E^2-\mathbf{p}^2=m_0^2\left[1+\frac{\|q\|^2}{8}\left(\frac{1}{\rho^2}-1\right)\right] \tag{B.7a}$$

Thus, we should define the four-momentum of a test particle in a different way, i.e., in the place of m_0 we should put

$$m_0^2\left[1+\frac{\|q\|^2}{2}\left(\frac{1}{\rho^2}-1\right)\right]^{1/2}=\bar{m}_0(\rho)$$

Equations (B.7) and (B.7a) give us a scalar field correction to the rest mass of a test particle. Now we can write some well-known formulas from special relativity generalized to this case:

$$E=\frac{\bar{m}_0(\rho)}{(1-V^2)^{1/2}} \tag{B.8}$$

$$\mathbf{P}=\frac{m_0(\rho)\mathbf{v}}{(1-V^2)^{1/2}} \tag{B.9}$$

where

$$\bar{m}_0(\rho)=m_0\left[1+\frac{\|q\|^2}{8}\left(\frac{1}{\rho^2}-1\right)\right]^{1/2} \tag{B.10}$$

and $\mathbf{v}$ is the velocity.

This gives us scalar field corrections to the famous Einstein formulas. In this way we get the scalar field-dependent rest mass of test particles. For the rest mass real we get the condition

$$|\rho|\leq\frac{\|q\|}{(\|q\|^2-8)^{1/2}} \tag{B.11}$$

and

$$\|q\|^2>8 \tag{B.12}$$

In terms of the scalar field Ψ one gets

$$\bar{m}_0(\Psi)=m_0\left[1+\frac{\|q\|}{8}(e^{2\Psi}-1)\right]^{1/2} \tag{B.13}$$

Thus, there is a maximum value of the scalar ρ (or Ψ) for which the special relativistic interpretation breaks down. This seems to be resonable because of the interpretation of the scalar field ρ (or Ψ). One has

$$G_{\text{eff}}=G_N\rho^{(n+2)}=G_N\,e^{-(n+2)\Psi} \tag{B.14}$$

This means that we have to deal with a very strong gravitational field, i.e.,

$$G_{\rm eff}^{\max} = G_N\left(\frac{\|q\|^2}{(\|q\|^2-8)^{1/2}}\right)^{n+2} \xrightarrow[\|q\|^2\to 8]{} \infty \tag{B.15}$$

For such a value of $G_{\rm eff}$ even the gravitational field of a test particle is strong. Now we go to an established frame with Cartesian coordinaties (x, y, z) and a time t. The equation of motion for a test particle can be written in the frame in the following way. Let us define

$$u^4 = \frac{dt}{d\tau} = \frac{[1+\frac{1}{8}\|q\|^2(1/\rho^2-1)]^{1/2}}{(1-v^2)^{1/2}} \tag{B.16}$$

$$\begin{aligned} u^j = \frac{dx^j}{d\tau} &= \frac{v^j}{(1-v^2)^{1/2}}\left[1+\frac{\|q\|^2}{8}\left(\frac{1}{\rho^2}-1\right)\right]^{1/2} \\ &= \frac{dx^j}{dt}\frac{[1+\frac{1}{8}\|q\|^2(1/\rho^2-1)]^{1/2}}{(1-v^2)^{1/2}} \end{aligned} \tag{B.17}$$

$j = 1, 2, 3$. Thus, we get from equation (B.2)

$$\begin{aligned} &\frac{d}{dt}\left[\mathbf{v}\left(\frac{1+\frac{1}{8}\|q\|^2(1/\rho^2-1)}{1-v^2}\right)^{1/2}\right] \\ &\quad = \left(\frac{q^a}{m_0}\right)\bar{g}_{ab}(\mathbf{E}^b + \mathbf{V}\times\mathbf{H}^b) \\ &\qquad + \frac{\|q\|^2}{8m_0^2}\left(\frac{1-v^2}{1+\frac{1}{8}\|q\|^2(1/\rho^2-1)}\right)^{1/2}\nabla\left(\frac{1}{\rho^2}\right) \end{aligned} \tag{B.18}$$

and

$$\begin{aligned} &\frac{d}{dt}\left[\left(\frac{1+\frac{1}{8}\|q\|^2(1/\rho^2-1)}{1-v^2}\right)^{1/2}\right] \\ &\quad = -\left(\frac{q^a}{m_0}\right)\bar{g}_{ab}\mathbf{E}^b\cdot\mathbf{v} \\ &\qquad + \frac{\|q\|^2}{8m_0^2}\left(\frac{1+\frac{1}{8}\|q\|^2(1/\rho^2-1)}{1-v^2}\right)^{1/2}\frac{d}{dt}\left(\frac{1}{\rho^2}\right) \end{aligned} \tag{B.19}$$

where

$$\bar{g}_{ab} = h_{ab} + 2\mu k_{ab} - \mu^2 k^d{}_a k_{ab} \tag{B.20}$$

$$\mathbf{E}^b = (H^b_{14}, H^b_{24}, H^b_{34}) \tag{B.21}$$

is the electric part of the Yang-Mills field and

$$\mathbf{B}^b = (H^b_{23}, H^b_{31}, H^b_{12}) \tag{B.22}$$

is the magnetic part of the Yang-Mills field. Let us define the kinetic energy of a test particle

$$E_K = E - \bar{m}_0(\rho) = \bar{m}_0(\rho)\left(\frac{1}{(1-v^2)^{1/2}} - 1\right) \tag{B.23}$$

Thus, the scalar field ρ (or Ψ) plays the role of the so-called rest mass field (see ref. 122), which is quite natural in scalar-tensor theories of gravitation.

Let us reconsider equation (14.25) for a zero m_0. One gets

$$\frac{du^\alpha}{d\tau} - v^a g_{ab} H^{ba}{}_\beta u^\beta - \frac{\|v\|^2}{8}\eta^{\alpha\beta}\left(\frac{1}{\rho^2}\right)_{,\beta} = 0 \tag{B.24}$$

$$\eta_{\alpha\beta}u^\alpha u^\beta = 0 \tag{B.24a}$$

where $v^a = \text{const}$ and is a measure of the coupling of a test particle to a gauge field. Thus, we consider an ultrarelativistic case, i.e., a massless "gluon" in a gauge field.

Equation (B.19) defines the change of the total energy of a test particle and it can be rewritten

$$\frac{dE}{dt} = -q^a g_{ab}\mathbf{E}^b \cdot \mathbf{v} + \frac{\|q\|^2}{8m_0^2}\left(\frac{1+\frac{1}{8}\|q\|^2(1/\rho^2-1)}{1-v^2}\right)^{1/2}\frac{d}{dt}\left(\frac{1}{\rho^2}\right) \tag{B.25}$$

Equation (B.18) can be rewritten in the form of a relativistic equation of motion in an established frame with Cartesian coordinates,

$$\frac{d\mathbf{p}}{dt} = q^a \bar{g}_{ab}(\mathbf{E}^b + \mathbf{v}\times\mathbf{B}^b) + \frac{\|q\|^2}{8m_0^2}\left(\frac{1-v^2}{1+\frac{1}{8}\|q\|^2(1/\rho^2-1)}\right)^{1/2}\nabla\left(\frac{1}{\rho^2}\right) \tag{B.26}$$

Let us take the nonrelativistic limit of (B.9) and (B.23) (i.e., small velocities)

$$E_k^{\text{nonrel}} = \frac{m_0}{2}\left[1 + \frac{\|q\|^2}{8}\left(\frac{1}{\rho^2} - 1\right)\right]^{1/2} v^2 \tag{B.27}$$

$$\mathbf{P}_{\text{nonrel}} = m_0\left[1 + \frac{\|q\|^2}{8}\left(\frac{1}{\rho^2} - 1\right)\right]^{1/2}\mathbf{v} \tag{B.28}$$

and equations (B.25), (B.26),

$$\frac{dE_k^{\text{nonrel}}}{dt} = +q^a \bar{g}_{ab}\mathbf{E}^b \mathbf{v} - \frac{\|q\|^2}{8m_0[1+\frac{1}{8}\|q\|^2(1/\rho^2-1)]^{1/2}}\frac{d}{dt}\left(\frac{1}{\rho^2}\right) \tag{B.29}$$

$$\frac{dp_{\text{nonrel}}}{dt} = -g^a \bar{g}_{ab}(\mathbf{E}^b + \mathbf{v}\times\mathbf{B}^b) - \frac{\|q\|^2}{8m_0[1+\frac{1}{8}\|q\|^2(1/\rho^2-1)]^{1/2}}\nabla\left(\frac{1}{\rho^2}\right) \tag{B.30}$$

In the electromagnetic case we get similarly

$$\frac{dE_k^{\text{nonrel}}}{dt} = -q\mathbf{E}\cdot\mathbf{v} + \frac{q^2}{8m}\frac{d}{dt}\left(\frac{1}{\rho^2}\right) \tag{B.31}$$

$$\frac{dp_{\text{nonrel}}}{dt} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) + \frac{q^2}{8m}\nabla\left(\frac{1}{\rho^2}\right) \tag{B.32}$$

where

$$m = m_0\left[1+\frac{\|q\|^2}{8}\left(\frac{1}{\rho^2}-1\right)\right]^{1/2}$$

E is the electric field and **B** is the magnetic field, and q is the electric charge of a test particle. Equations (B.30) and (B.32) can be considered as the fifth force correction to the nonrelativistic motion of charged test bodies. Simultaneously we get that the inertial (nonrelativistic) mass depends on the scalar field ρ (or Ψ) and this dependence is "chemical composition" dependent,

$$m_{\text{nonrel}} = m_0\left[1+\frac{\|q\|^2}{8}\left(\frac{1}{\rho^2}-1\right)\right]^{1/2} \tag{B.33}$$

The mass m_0, because of equation (B.1), has the interpretation of the gravitational charge of a test particle. This means that it is a passive gravitational mass. Thus, one gets

$$\frac{m_{\text{inertial}}}{m_{\text{grav.passive}}} = \left[1+\frac{\|q\|^2}{8}\left(\frac{1}{\rho^2}-1\right)\right]^{1/2} \tag{B.34}$$

In this approach the law of gravity is universal, but the inertia depends on the composition of a body. This is exactly the reverse of other approaches. Thus, the scalar field $\rho(\Psi)$ plays the role of a "rest mass field."

Let us calculate the scalar field correction to the inertial mass of a charged particle,

$$m_{\text{inertial}} = m_0 + \Delta m(\rho) \cong m_0 + \frac{\|q\|^2}{8}\Delta\rho m_0 \tag{B.35}$$

where

$$\rho = 1 + \Delta\rho$$

and

$$|\Delta\rho| \ll 1$$

On the other hand, we have

$$G_{\text{eff}} = G_N + \Delta G \cong G_N(1+(n+2)\Delta\rho) \tag{B.36}$$

Supposing that the field ρ does not change much spatially, we can derive a small correction to the Lorentz force in (B.30),

$$\frac{d\mathbf{v}}{dt}=\frac{q^a}{m_0}\bar{g}_{ab}(\mathbf{E}^b+\mathbf{v}\times\mathbf{B})+\|q\|^2\left(\frac{\Delta\rho}{8}\right)\frac{d\mathbf{v}}{dt} \tag{B.37}$$

This correction can be tested experimentally because it is easy to extract it from the ordinary Lorentz force. This is possible due to the different dependence on q and because of the appearance of $d\mathbf{v}/dt$. For zero electric and magnetic fields on the surface of the earth we get

$$\mathbf{a}=\frac{1}{1+\frac{1}{8}\|q\|^2\,\Delta\rho}\,\mathbf{g} \tag{B.38}$$

where $\mathbf{g}$ is the gravitational acceleration on the surface of the earth. Taking two samples with different charges

$$\|q_i\|^2=r_i, \qquad i=1,2$$

one can measure the difference in accelerations

$$\Delta a=a_1-a_2\cong\frac{\Delta r}{8}\,\Delta\rho g \tag{B.39}$$

where

$$\Delta r=\|q_2\|^2-\|q_1\|^2$$

Thus, we get

$$\left(\frac{\Delta a}{g}\right)=\frac{\Delta\rho}{8}\,\Delta r \tag{B.40}$$

This means a linear dependence. Thus, measuring differences in the acceleration of pairs of samples, we can test the predictions of the theory. For example, we can try to reinterpret in a different way the results of Fishbach *et al.* (see ref. 54) on a reanalysis of the Eötvös experiment.

Let us reconsider equation (14.28a) in a static field. Thus,

$$ds^2=g_{44}(dx^4)^2-dl^2=g_{44}(dx^4)^2-\gamma_{\bar{i}\bar{j}}\,dx^{\bar{i}}\,dx^{\bar{j}}, \qquad \bar{i},\bar{j}=1,2,3$$

and g_{44}, $\gamma_{\bar{a}\bar{b}}$ do not depend on x^4. One gets similarly to equations (B.7a) and (B.8)

$$E^2-P_{\bar{K}}P^{\bar{K}}=m_0^2\left[1+\frac{\|q\|^2}{8}\left(\frac{1}{\rho^2}-1\right)\right]=\bar{m}_0^2(\rho) \tag{B.41}$$

where

$$E = \bar{m}_0(\rho) g_{44} \frac{dx^4}{ds}$$

$$= \bar{m}_0(\rho) g_{44} \frac{dx^4}{[g_{44}(dx^4)^2 - dl^2]^{1/2}}$$

$$= \frac{\bar{m}_0(\rho)(g_{44})^{1/2}}{(1-v^2)^{1/2}} \tag{B.42}$$

$$v = \frac{dl}{d\tau} = \frac{dl}{(g_{44})^{1/2}\, dx^4} \tag{B.43}$$

$$p_{\bar{k}} = \bar{m}_0(\rho) u_{\bar{k}} = \bar{m}_0(\rho) \gamma_{\bar{k}\bar{i})} u^{\bar{i}} \tag{B.44}$$

$$P_{\bar{k}} P^{\bar{k}} = m_0^2(\rho) \gamma_{(\bar{i}\bar{j})} u^{\bar{i}} u^{\bar{j}} \tag{B.45}$$

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